

ON THE COMPUTATION OF RATIONAL POINTS OF A HYPERSURFACE OVER A FINITE FIELD

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ABSTRACT. We design and analyze an algorithm for computing rational points of hypersurfaces defined over a finite field based on searches on “vertical strips”, namely searches on parallel lines in a given direction. Our results show that, on average, less than two searches suffice to obtain a rational point. We also analyze the probability distribution of outputs, using the notion of Shannon entropy, and prove that the algorithm is somewhat close to any “ideal” equidistributed algorithm.

1. INTRODUCTION

Let \mathbb{F}_q be the finite field of q elements, X_1, \dots, X_r indeterminates over \mathbb{F}_q and $\mathbb{F}_q[X_1, \dots, X_r]$ the ring of polynomials in X_1, \dots, X_r with coefficients in \mathbb{F}_q . Let $\mathcal{F}_{r,d} := \{F \in \mathbb{F}_q[X_1, \dots, X_r] : \deg(F) \leq d\}$. Suppose that $r \geq 2$ and $d \geq 2$, and let F be an element of $\mathcal{F}_{r,d}$. In this paper we address the problem of finding an \mathbb{F}_q -rational zero of F , namely a point $\mathbf{x} \in \mathbb{F}_q^r$ with $F(\mathbf{x}) = 0$.

It is well-known that the elements of $\mathcal{F}_{r,d}$ have q^{r-1} zeros in \mathbb{F}_q^r on average. More precisely, we have the following result (see, e.g., [19, Theorem 6.16]):

$$(1.1) \quad \frac{1}{|\mathcal{F}_{r,d}|} \sum_{F \in \mathcal{F}_{r,d}} N(F) = q^{r-1},$$

where $N(F) := |\{\mathbf{x} \in \mathbb{F}_q^r : F(\mathbf{x}) = 0\}|$. This suggests a strategy to find an \mathbb{F}_q -rational zero of a given $F \in \mathcal{F}_{r,d}$. Since the expected number of zeros of F is equal to the cardinality of \mathbb{F}_q^{r-1} , given $\mathbf{a}_1 \in \mathbb{F}_q^{r-1}$, one may try to find a zero of F having \mathbf{a}_1 as its first $r-1$ coordinates. If the polynomial $F(\mathbf{a}_1, X_r)$ has no zeros in \mathbb{F}_q , then a further element $\mathbf{a}_2 \in \mathbb{F}_q^{r-1}$ can be picked up to see whether $F(\mathbf{a}_2, X_r)$ has a zero in \mathbb{F}_q . The algorithm proceeds in this way until a zero of F in \mathbb{F}_q^r is obtained.

Following the terminology of [15], which considers the case $r = 2$, each set $\{\mathbf{a}_i\} \times \mathbb{F}_q$ is called a “vertical strip”. Therefore, our algorithm, which extends the one of [15] to r -variate polynomials, is called “Search on Vertical Strips” (SVS for short), and is described as follows.

Algorithm SVS.

Input: a polynomial $F \in \mathcal{F}_{r,d}$.

Output: either a zero $\mathbf{x} \in \mathbb{F}_q^r$ of F , or “failure”.

Set $i := 1$ and $f := 1$

While $1 \leq i \leq q^{r-1}$ and $f = 1$ do

Choose at random $\mathbf{a}_i \in \mathbb{F}_q^{r-1} \setminus \{\mathbf{a}_1, \dots, \mathbf{a}_{i-1}\}$

Compute $f := \gcd(F(\mathbf{a}_i, X_r), X_r^q - X_r)$

If $f = 0$, then choose $x_{r,i} \in \mathbb{F}_q$ at random

If $f \notin \{0, 1\}$, then compute a root $x_{r,i} \in \mathbb{F}_q$ of f

$i := i + 1$

Date: November 21, 2016.

1991 *Mathematics Subject Classification.* 68W40, 11G25, 14G05, 14G15.

Key words and phrases. Finite fields, hypersurfaces, rational points, algorithms, average-case complexity, probability distribution, value sets, Shannon entropy.

The authors were partially supported by the grants PIP CONICET 11220130100598, PIO conicet-ungs 14420140100027 and UNGS 30/3084.

End While

If $f \neq 1$ return $(\mathbf{a}_i, x_{r,i})$, else return “failure”.

Ignoring the cost of random generation of elements of \mathbb{F}_q^{r-1} , at the i th step of the main loop we compute the vector of coefficients of the polynomial $F(\mathbf{a}_i, X_r)$. Since an element of $\mathcal{F}_{r,d}$ has $D := \binom{d+r}{r}$ coefficients, the number of arithmetic operations in \mathbb{F}_q required to compute such a vector is $\mathcal{O}^\sim(D)$, where the notation \mathcal{O}^\sim ignores logarithmic factors. Throughout this paper, all asymptotic estimates are valid for fixed d and r , and q growing to infinity. Then the gcd f is computed, and a root of f in \mathbb{F}_q is determined, provided that $f \neq 1$. This can be done with $\mathcal{O}^\sim(d \log_2 q)$ arithmetic operations in \mathbb{F}_q (see, e.g., [14, Corollary 14.16]). As a consequence, for a choice $\underline{\mathbf{a}} := (\mathbf{a}_1, \dots, \mathbf{a}_{q^{r-1}})$ for the vertical strips to be considered, the whole procedure requires $\mathcal{O}^\sim(C_{\underline{\mathbf{a}}}(F) \cdot (D + d \log_2 q))$ arithmetic operations in \mathbb{F}_q , where $C_{\underline{\mathbf{a}}}(F)$ is the least value of i for which $F(\mathbf{a}_i, X_r)$ has a zero in \mathbb{F}_q .

This paper is devoted to analyze the SVS algorithm from a probabilistic point of view. As its behavior is essentially determined by the number of vertical strips which must be considered, we analyze, for a given $s \geq 1$, the probability distribution of the number of searches performed by the algorithm. For this purpose, we consider the set \mathbf{F} of all possible choices of vertical strips and the random variable $C_{r,d} : \mathbf{F} \times \mathcal{F}_{r,d} \mapsto \mathbb{N}$ which counts the number of vertical strips that are searched. We prove that the probability that s vertical strips are searched, for “moderate” values of s , satisfies the estimate

$$(1.2) \quad P[C_{r,d} = s] = (1 - \mu_d)^{s-1} \mu_d + \mathcal{O}(q^{-1/2}),$$

where $\mu_d := \sum_{j=1}^d (-1)^{j-1} / j!$. Observe that $\mu_d \approx 1 - e^{-1} = 0.6321 \dots$ for large d , where e denotes the basis of the natural logarithm. We remark that the quantity μ_d arises also in connection with a classical combinatorial notion over finite fields, that of the value set of univariate polynomials (cf. [19], [23]). For a polynomial $f \in \mathbb{F}_q[T]$, denote by $\mathcal{V}(f) := |\{f(c) : c \in \mathbb{F}_q\}|$ the cardinality of the value set of f . In [4], Birch and Swinnerton-Dyer established the following classical result: if $f \in \mathbb{F}_q[T]$ is a generic polynomial of degree d , then $\mathcal{V}(f) = \mu_d q + \mathcal{O}(1)$.

The estimate (1.2) relies on the analysis of the behavior of the SVS algorithm for a fixed choice $\mathbf{a}_1, \dots, \mathbf{a}_s \in \mathbb{F}_q^{r-1}$ for the first s vertical strips. It turns out that the probability that the s vertical strips under consideration are searched is essentially that of the right-hand side of (1.2). As a side note, this may be considered as a “realistic” version of the SVS algorithm in the sense of [1]. As the author states, “when a randomized algorithm is implemented, one always uses a sequence whose later values come from earlier ones in a deterministic fashion. This invalidates the assumption of independence and might cause one to regard results about probabilistic algorithms with suspicion.” Our results show that the probabilistic behavior of the SVS algorithm is not essentially altered when a fixed choice of vertical strips is considered.

As a consequence of (1.2) we obtain an upper bound on the average-case complexity $E[X]$ of the SVS algorithm, where $X : \mathbf{F} \times \mathcal{F}_{r,d} \rightarrow \mathbb{N}$ is the random variable that counts the number of arithmetic operations in \mathbb{F}_q performed for a given choice of vertical strips on a given input. We prove that

$$(1.3) \quad E[X] \leq \frac{1}{\mu_d} \tau(d, r, q) + \mathcal{O}(q^{-1/2}),$$

where $\tau(d, r, q) := \mathcal{O}^\sim(D + d \log_2 q)$ is the cost of a search in a single vertical strip. In other words, on average at most $1/\mu_d \approx 1.58$ vertical strips must be searched to obtain a rational zero of the polynomial under consideration. Simulations we run suggest that the upper bound (1.3) is close to optimal. We observe that the probabilistic algorithms of [15] (for $r = 2$) and [5] and [20] (for general r) propose d searches in order to achieve a probability of success greater than $1/2$. Our result suggests that these analyses are somewhat pessimistic.

On the other hand, it must be said that the result of [15] holds for any bivariate polynomial, while that of [5] is valid for any absolutely irreducible r -variate polynomial. If the polynomials under consideration are produced by some complicated process, it might be argued that our results do not contribute to the analysis of the cost of the corresponding algorithm to search for \mathbb{F}_q -rational zeros. Nevertheless, a crucial aspect of our approach is that we express the probability $P[C_{r,d} = s]$ of (1.2), and thus the average-case complexity $E[X]$ of (1.3), in terms of the average cardinality of the value set of certain families of univariate polynomials related to the set of input polynomials under consideration. We believe that this technique can be extended to deal with (linear or nonlinear) families of polynomials of $\mathcal{F}_{r,d}$, provided that the asymptotic behavior of the average cardinality of the corresponding families of univariate polynomials is known (see [8], [21] and [22] for results in connection with this matter).

Another critical aspect to analyze is the distribution of outputs. Given $F \in \mathcal{F}_{r,d}$, the SVS algorithm outputs an \mathbb{F}_q -rational zero of F , which is determined by certain random choices made during its execution. As a consequence, it is relevant to have insight on the probability distribution of outputs. For an “ideal” algorithm (from the point of view of distribution of outputs), outputs should be equidistributed. For this reason, in [15] the basic SVS strategy for bivariate polynomials over \mathbb{F}_q is modified so that all \mathbb{F}_q -rational zeros of the input polynomial are equally probable outputs. Such a modification can be also be applied to our algorithm.

Nevertheless, as this modification implies a certain slowdown, we shall pursue a different course of action, analyzing the average distribution of outputs by means of the concept of Shannon entropy. If the output for an input polynomial F tends to be concentrated on a few \mathbb{F}_q -rational zeros of F , then the “amount of information” that we obtain might be said to be “small”. On the other hand, if all the \mathbb{F}_q -rational zeros of F are equally probable outputs, then the amount of information provided by the algorithm is considered to be larger. Following [3] (see also [2]), we define a Shannon entropy H_F associated to an input $F \in \mathcal{F}_{r,d}$ of the SVS algorithm, which measures how “concentrated” are the corresponding outputs. Then we analyze the average entropy H when F runs through all the elements of $\mathcal{F}_{r,d}$.

For an “ideal” algorithm for computing \mathbb{F}_q -rational zeros of elements of $\mathcal{F}_{r,d}$ and $F \in \mathcal{F}_{r,d}$, it is easy to see that $H_F^{\text{ideal}} = \log N(F)$, where \log denotes the natural logarithm. It follows that

$$H^{\text{ideal}} \leq \log(q^{r-1})$$

(see (5.3)). Our main result concerning the distribution of outputs asserts that

$$(1.4) \quad H \geq \frac{1}{2\mu_d} \log(q^{r-1})(1 + \mathcal{O}(q^{-1})).$$

Since $1/2\mu_d \approx 0.79$ for large d , we may paraphrase (1.4) as saying that the SVS algorithm is at least 79 per cent as good as any “ideal” algorithm, from the point of view of the distribution of the outputs.

The proof of (1.4) relies on an analysis of the expected number of vertical strips of the elements of $\mathcal{F}_{r,d}$ which may be of independent interested. Denote by $NS(r, d)$ the average number of vertical strips with \mathbb{F}_q -rational zeros of F , when F runs through all the elements of $\mathcal{F}_{r,d}$. We prove that

$$(1.5) \quad NS(r, d) = \mu_d q^{r-1} + \mathcal{O}(q^{r-2}).$$

We also estimate the variance of the number of vertical strips with \mathbb{F}_q -rational zeros.

The paper is organized as follows. Section 2 is devoted to the analyses of the probability that one or two vertical strips are searched. In Section 3 we estimate the expected number of vertical strips to be searched for a given choice of $s \geq 3$ vertical strips. We express the probability that s vertical strips are searched in terms of average cardinalities of value sets and apply estimates for the latter in order to establish an explicit estimate of the former. In Section 4 we apply the results of Sections 2 and 3 to establish (1.2) and (1.3). Section 5 is concerned with the probability distribution of outputs. In Subsection 5.1 we establish (1.5)

and an estimate of the corresponding variance. In Subsection 5.2 we apply these estimates to prove (1.4). Finally, in Section 6 we exhibit a few simulations aimed at confirming the asymptotic results (1.2) and (1.3).

2. PROBABILITY OF SUCCESS IN THE FIRST TWO SEARCHES

We start discussing how frequently one or two searches on vertical strips suffice to find a zero of the input polynomial. As it will become evident, this will happen in most cases. Therefore, accurate estimates on the probability of these two cases is critical for an accurate description of the behavior of the algorithm.

2.1. Probability of success in the first search. For integers $r \geq 2$ and $d \geq 2$, we shall estimate the probability that the SVS algorithm, on input an element of $\mathcal{F}_{r,d} := \{F \in \mathbb{F}_q[X_1, \dots, X_r] : \deg(F) \leq d\}$, finds a root of it in the first vertical strip. As r and d are fixed, we shall drop the indices r and d from the notations.

Each possible choice for the first vertical strip is determined by an element of \mathbb{F}_q^{r-1} . As a consequence, we may represent the situation by means of the random variable $C_1 := C_{1,r,d} : \mathbb{F}_q^{r-1} \times \mathcal{F}_{r,d} \rightarrow \{1, \infty\}$ defined in the following way:

$$C_1(\mathbf{a}, F) := \begin{cases} 1 & \text{if } F(\mathbf{a}, X_r) \text{ has an } \mathbb{F}_q\text{-rational zero,} \\ \infty & \text{otherwise.} \end{cases}$$

We consider the set $\mathbb{F}_q^{r-1} \times \mathcal{F}_{r,d}$ endowed with the uniform probability $P_1 := P_{1,r,d}$ and study the probability of the set $\{C_1 = 1\}$. The next result provides an exact formula for this probability.

Theorem 2.1. *For $q > d$, we have the identity*

$$P_1[C_1 = 1] = \sum_{j=1}^d (-1)^{j-1} \binom{q}{j} q^{-j} + (-1)^d \binom{q-1}{d} q^{-d-1}.$$

Proof. For any $F \in \mathcal{F}_{r,d}$, we denote by $VS(F)$ the set of vertical strips where F has an \mathbb{F}_q -rational zero and by $NS(F)$ its cardinality, that is,

$$VS(F) := \{\mathbf{a} \in \mathbb{F}_q^{r-1} : (\exists x_r \in \mathbb{F}_q) F(\mathbf{a}, x_r) = 0\}, \quad NS(F) := |VS(F)|.$$

It is easy to see that $\{C_1 = 1\} = \bigcup_{F \in \mathcal{F}_{r,d}} VS(F) \times \{F\}$. Since this is a union of disjoint subsets of $\mathbb{F}_q^{r-1} \times \mathcal{F}_{r,d}$, it follows that

$$(2.1) \quad P_1[C_1 = 1] = \frac{1}{q^{r-1}|\mathcal{F}_{r,d}|} \sum_{F \in \mathcal{F}_{r,d}} NS(F).$$

Fix $F \in \mathcal{F}_{r,d}$. Observe that

$$VS(F) = \bigcup_{x \in \mathbb{F}_q} \{\mathbf{a} \in \mathbb{F}_q^{r-1} : F(\mathbf{a}, x) = 0\}.$$

As a consequence, by the inclusion–exclusion principle we obtain

$$\begin{aligned} NS(F) &= \left| \bigcup_{x \in \mathbb{F}_q} \{\mathbf{a} \in \mathbb{F}_q^{r-1} : F(\mathbf{a}, x) = 0\} \right| \\ &= \sum_{j=1}^q (-1)^{j-1} \sum_{\mathcal{X}_j \subset \mathbb{F}_q} |\{\mathbf{a} \in \mathbb{F}_q^{r-1} : (\forall x \in \mathcal{X}_j) F(\mathbf{a}, x) = 0\}|, \end{aligned}$$

where \mathcal{X}_j runs through all the subsets of \mathbb{F}_q of cardinality j . We conclude that

$$\sum_{F \in \mathcal{F}_{r,d}} NS(F) = \sum_{F \in \mathcal{F}_{r,d}} \sum_{j=1}^q (-1)^{j-1} \sum_{\mathcal{X}_j \subset \mathbb{F}_q} |\{\mathbf{a} \in \mathbb{F}_q^{r-1} : (\forall x \in \mathcal{X}_j) F(\mathbf{a}, x) = 0\}|.$$

For any j with $1 \leq j \leq q$, we denote

$$\mathcal{N}_j := \frac{1}{q^{r-1}|\mathcal{F}_{r,d}|} \sum_{F \in \mathcal{F}_{r,d}} \sum_{\mathcal{X}_j \subset \mathbb{F}_q} |\{\mathbf{a} \in \mathbb{F}_q^{r-1} : (\forall x \in \mathcal{X}_j) F(\mathbf{a}, x) = 0\}|,$$

where \mathcal{X}_j runs through all the subsets of \mathbb{F}_q of cardinality j . If $j \leq d$ and \mathbf{a} is fixed, then the equalities $F(\mathbf{a}, x) = 0$ ($x \in \mathcal{X}_j$) are j linearly-independent conditions on the coefficients of F in the \mathbb{F}_q -vector space $\mathcal{F}_{r,d}$. It follows that

$$\begin{aligned} \mathcal{N}_j &= \frac{1}{q^{r-1}|\mathcal{F}_{r,d}|} \sum_{\mathcal{X}_j \subset \mathbb{F}_q} \sum_{\mathbf{a} \in \mathbb{F}_q^{r-1}} |\{F \in \mathcal{F}_{r,d} : (\forall x \in \mathcal{X}_j) F(\mathbf{a}, x) = 0\}| \\ (2.2) \quad &= \frac{1}{q^{r-1+\dim \mathcal{F}_{r,d}}} \sum_{\mathcal{X}_j \subset \mathbb{F}_q} \sum_{\mathbf{a} \in \mathbb{F}_q^{r-1}} q^{\dim \mathcal{F}_{r,d-j}} = \binom{q}{j} q^{-j}. \end{aligned}$$

On the other hand, if $j > d$, then $F(\mathbf{a}, x) = 0$ for every $x \in \mathcal{X}_j$ if and only if $F(\mathbf{a}, X_r) = 0$. The condition $F(\mathbf{a}, X_r) = 0$ is expressed by means of $d+1$ linearly-independent linear equations on the coefficients of F in $\mathcal{F}_{r,d}$. We conclude that

$$(2.3) \quad \mathcal{N}_j = \frac{1}{q^{r-1+\dim \mathcal{F}_{r,d}}} \sum_{\mathcal{X}_j \subset \mathbb{F}_q} \sum_{\mathbf{a} \in \mathbb{F}_q^{r-1}} q^{\dim \mathcal{F}_{r,d-(d+1)}} = \binom{q}{j} q^{-d-1}.$$

Combining (2.2) and (2.3) we obtain

$$P_1[C_1 = 1] = \sum_{j=1}^q (-1)^{j-1} \mathcal{N}_j = \sum_{j=1}^d (-1)^{j-1} \binom{q}{j} q^{-j} + \sum_{j=d+1}^q (-1)^{j-1} \binom{q}{j} q^{-d-1}.$$

Finally, since

$$(2.4) \quad \sum_{j=d+1}^q (-1)^{j-1} \binom{q}{j} = \sum_{j=0}^d (-1)^j \binom{q}{j} = (-1)^d \binom{q-1}{d}$$

(see, e.g., [17, (5.16)]), we readily deduce the statement of the theorem. \square

Next we discuss the asymptotic behavior of the probability $P_1[C_1 = 1]$. Fix $d \geq 2$. From Theorem 2.1 it can be seen that

$$P_1[C_1 = 1] = \mu_d + \mathcal{O}(q^{-1}), \quad \mu_d := \sum_{j=1}^d \frac{(-1)^{j-1}}{j!}.$$

To show this, given positive integers k, j with $k \leq j$, we shall denote by $\begin{bmatrix} j \\ k \end{bmatrix}$ the unsigned Stirling number of the first kind, namely the number of permutations of j elements with k disjoint cycles. The following properties of the Stirling numbers are well-known (see, e.g., [13, §A.8]):

$$\begin{bmatrix} j \\ j \end{bmatrix} = 1, \quad \begin{bmatrix} j \\ j-1 \end{bmatrix} = \binom{j}{2}, \quad \sum_{k=0}^j \begin{bmatrix} j \\ k \end{bmatrix} = j!.$$

We shall also use the following well-known identity (see, e.g., [17, (6.13)]):

$$(2.5) \quad \binom{q}{j} = \sum_{k=0}^j \frac{(-1)^{j-k}}{j!} \begin{bmatrix} j \\ k \end{bmatrix} q^k.$$

According to Theorem 2.1 and (2.5), we have

$$\begin{aligned} P_1[C_1 = 1] &= \sum_{j=1}^d (-1)^{j-1} \sum_{k=0}^j \frac{(-1)^{j-k}}{j!} \begin{bmatrix} j \\ k \end{bmatrix} q^{k-j} + (-1)^d \binom{q-1}{d} q^{-d-1} \\ &= \sum_{j=1}^d \frac{(-1)^{j-1}}{j!} \begin{bmatrix} j \\ j \end{bmatrix} + \sum_{j=1}^d \frac{(-1)^j}{j!} \begin{bmatrix} j \\ j-1 \end{bmatrix} q^{-1} \\ &\quad + \sum_{j=1}^d \sum_{k=0}^{j-2} \frac{(-1)^{k-1}}{j!} \begin{bmatrix} j \\ k \end{bmatrix} q^{k-j} + (-1)^d \binom{q-1}{d} q^{-d-1}. \end{aligned}$$

It follows that

$$P_1[C_1 = 1] = \mu_d + \frac{1}{q} \sum_{j=1}^d \frac{(-1)^j}{j!} \binom{j}{2} - \sum_{j=1}^d \sum_{k=0}^{j-2} \frac{(-1)^k}{j!} \left[\begin{matrix} j \\ k \end{matrix} \right] q^{k-j} + \frac{(-1)^d}{q^{d+1}} \binom{q-1}{d}.$$

As a consequence, for $d > 2$ we obtain

$$\begin{aligned} |P_1[C_1 = 1] - \mu_d| &\leq \frac{1}{q} \left| \sum_{j=1}^d \frac{(-1)^j}{j!} \binom{j}{2} \right| + \sum_{j=1}^d \sum_{k=0}^{j-2} \frac{1}{j!} \left[\begin{matrix} j \\ k \end{matrix} \right] \frac{1}{q^2} + \frac{1}{q^{d+1}} \binom{q-1}{d} \\ &\leq \frac{1}{4q} + \frac{d}{q^2} + \frac{1}{2q}. \end{aligned}$$

For $d = 2$, this inequality is obtained by a direct calculation. We have therefore the following result.

Corollary 2.2. *For $q > d$,*

$$|P_1[C_1 = 1] - \mu_d| \leq \frac{2}{q}.$$

As d tends to infinity, the number $P_1[C_1 = 1]$ tends to $1 - e^{-1} = 0.6321\dots$, where e denotes the basis of the natural logarithm. This explains the numerical results in the first row of the tables of the simulations of Section 6.

It is worth remarking that the quantity $P_1[C_1 = 1]$ is closely connected with the probability that a univariate polynomial of degree at most d has \mathbb{F}_q -rational roots. More precisely, consider the set $\mathcal{F}_{1,d}$ of univariate polynomials of degree at most d with coefficients in \mathbb{F}_q , endowed with the uniform probability $p_{1,d}$, and the random variable $N_{1,d} : \mathcal{F}_{1,d} \rightarrow \mathbb{Z}_{\geq 0}$ which counts the number of \mathbb{F}_q -rational zeros, namely

$$N_{1,d}(f) := |\{x \in \mathbb{F}_q : f(x) = 0\}|.$$

The random variable $N_{1,d}$ has been implicitly studied in the literature (see, e.g., [9, §2] or [18, Theorem 3]). It can be proved that, for $q > d$,

$$p_{1,d}[N_{1,d} > 0] = P_1[C_1 = 1].$$

2.2. Probability of success in the second search. Next we analyze the probability that the SVS algorithm performs exactly two searches.

Each possible choice for the first two vertical strips is determined by an element $\underline{a} := (\mathbf{a}_1, \mathbf{a}_2) \in \mathbb{F}_q^{r-1} \times \mathbb{F}_q^{r-1}$ with $\mathbf{a}_1 \neq \mathbf{a}_2$. Therefore, we denote by \mathbf{F}_2 the set of all such possible choices and by N_2 its cardinality, that is,

$$\mathbf{F}_2 := \{\underline{a} := (\mathbf{a}_1, \mathbf{a}_2) \in \mathbb{F}_q^{r-1} \times \mathbb{F}_q^{r-1} : \mathbf{a}_1 \neq \mathbf{a}_2\}, \quad N_2 = |\mathbf{F}_2| = q^{r-1}(q^{r-1} - 1).$$

We shall study the random variable $C_2 := C_{2,r,d} : \mathbf{F}_2 \times \mathcal{F}_{r,d} \rightarrow \{1, 2, \infty\}$ defined as

$$C_2(\underline{a}, F) := \begin{cases} 1 & \text{if } N_{1,d}(F(\mathbf{a}_1, X_r)) > 0, \\ 2 & \text{if } N_{1,d}(F(\mathbf{a}_1, X_r)) = 0 \text{ and } N_{1,d}(F(\mathbf{a}_2, X_r)) > 0, \\ \infty & \text{otherwise.} \end{cases}$$

We consider the set $\mathbf{F}_2 \times \mathcal{F}_{r,d}$ endowed with the uniform probability $P_2 := P_{2,r,d}$. We aim to determine the probability $P_2[C_2 = 2]$.

This probability will be expressed in terms of probabilities concerning the random variables $C_{\underline{a}} := C_{\underline{a},r,d} : \mathcal{F}_{r,d} \rightarrow \{1, 2, \infty\}$ which count the number of searches that are performed on the vertical strips defined by $\underline{a} := (\mathbf{a}_1, \mathbf{a}_2) \in \mathbf{F}_2$ until an \mathbb{F}_q -rational zero is obtained, $C_{\underline{a}}(F) = \infty$ meaning that F does not have \mathbb{F}_q -rational zeros on these two vertical strips. For this purpose, the set $\mathcal{F}_{r,d}$ is considered to be endowed with the uniform probability $p_{r,d}$. The relation between these random variables and $P_2[C_2 = 2]$ is expressed in the following lemma.

Lemma 2.3. *We have*

$$P_2[C_2 = 2] = \frac{1}{N_2} \sum_{\underline{a} \in \mathbf{F}_2} p_{r,d}[C_{\underline{a}} = 2].$$

Proof. Observe that

$$\{C_2 = 2\} = \bigcup_{\underline{a} \in \mathbb{F}_2} \{\underline{a}\} \times \{F \in \mathcal{F}_{r,d} : C_{\underline{a}}(F) = 2\}.$$

Since this is union of disjoint sets, we conclude that

$$P_2[C_2 = 2] = \frac{1}{N_2} \sum_{\underline{a} \in \mathbb{F}_2} \frac{|\{F \in \mathcal{F}_{r,d} : C_{\underline{a}}(F) = 2\}|}{|\mathcal{F}_{r,d}|} = \frac{1}{N_2} \sum_{\underline{a} \in \mathbb{F}_2} p_{r,d}[C_{\underline{a}} = 2],$$

which proves the lemma. \square

Next we estimate the probability $p_{r,d}[C_{\underline{a}} = 2]$ for a given $\underline{a} \in \mathbb{F}_2$.

Proposition 2.4. *For $q > d$ and $\underline{a} := (\mathbf{a}_1, \mathbf{a}_2) \in \mathbb{F}_2$, we have*

$$|p_{r,d}[C_{\underline{a}} = 2] - \mu_d(1 - \mu_d)| \leq \frac{3}{q}.$$

Proof. Observe that

$$\{C_{\underline{a}} = 2\} = \{F \in \mathcal{F}_{r,d} : N_{1,d}(F(\mathbf{a}_2, T)) > 0\} \setminus \{F \in \mathcal{F}_{r,d} : N_{1,d}(F(\mathbf{a}_1, T)) > 0\}.$$

The number of elements of $\mathcal{F}_{r,d}$ having \mathbb{F}_q -rational zeros in the vertical strip defined by \mathbf{a}_2 is determined in Theorem 2.1. Therefore, it remains to find the number $N_{\underline{a},2}$ of elements of $\mathcal{F}_{r,d}$ having \mathbb{F}_q -rational zeros both in the vertical strips defined by \mathbf{a}_1 and \mathbf{a}_2 . We have

$$N_{\underline{a},2} = \left| \bigcup_{x \in \mathbb{F}_q} \bigcup_{y \in \mathbb{F}_q} \{F \in \mathcal{F}_{r,d} : F(\mathbf{a}_1, x) = F(\mathbf{a}_2, y) = 0\} \right|.$$

Given sets $\mathcal{X} \subset \mathbb{F}_q$ and $\mathcal{Y} \subset \mathbb{F}_q$, we denote

$$\mathcal{S}_{\underline{a}}(\mathcal{X}, \mathcal{Y}) := \{F \in \mathcal{F}_{r,d} : F(\mathbf{a}_1, x) = F(\mathbf{a}_2, y) = 0 \text{ for all } x \in \mathcal{X} \text{ and } y \in \mathcal{Y}\}.$$

Then the inclusion-exclusion principle implies

$$(2.6) \quad N_{\underline{a},2} = \sum_{j=1}^q \sum_{k=1}^q (-1)^{j+k} \sum_{\mathcal{X}_j \subset \mathbb{F}_q} \sum_{\mathcal{Y}_k \subset \mathbb{F}_q} |\mathcal{S}_{\underline{a}}(\mathcal{X}_j, \mathcal{Y}_k)|.$$

where the sum runs over all subsets $\mathcal{X}_j \subset \mathbb{F}_q$ and $\mathcal{Y}_k \subset \mathbb{F}_q$ of j and k elements respectively.

Claim. $\frac{N_{\underline{a},2}}{|\mathcal{F}_{r,d}|} = (P_1[C_1 = 1])^2 + \frac{q-1}{q^{2d+2}} \binom{q-1}{d}^2 = (P_1[C_1 = 1])^2 + \mathcal{O}(q^{-1}).$

Proof of Claim. For $1 \leq j, k \leq q$, let

$$\mathcal{N}_{j,k} := \sum_{\mathcal{X}_j \subset \mathbb{F}_q} \sum_{\mathcal{Y}_k \subset \mathbb{F}_q} |\mathcal{S}_{\underline{a}}(\mathcal{X}_j, \mathcal{Y}_k)|.$$

We determine $\mathcal{N}_{j,k}$ according to whether one of the following four cases occurs.

First suppose that $j, k \leq d$. As $\mathbf{a}_1 \neq \mathbf{a}_2$, the equalities $F(\mathbf{a}_1, x) = 0, F(\mathbf{a}_2, y) = 0$ for all $x \in \mathcal{X}_j$ and $y \in \mathcal{Y}_k$ impose $j + k$ linearly-independent conditions on the coefficients of $F \in \mathcal{F}_{r,d}$. Therefore, $|\mathcal{S}_{\underline{a}}(\mathcal{X}_j, \mathcal{Y}_k)| = q^{\dim \mathcal{F}_{r,d} - j - k}$, which implies

$$\mathcal{N}_{j,k} = \sum_{\mathcal{X}_j \subset \mathbb{F}_q} \sum_{\mathcal{Y}_k \subset \mathbb{F}_q} q^{\dim \mathcal{F}_{r,d} - j - k} = \binom{q}{j} \binom{q}{k} q^{\dim \mathcal{F}_{r,d} - j - k}.$$

The second case is determined by the conditions $j > d$ and $k \leq d$. If $j > d$ and $\mathcal{X}_j \subset \mathbb{F}_q$ is a subset of cardinality j , then the condition $F(\mathbf{a}_1, x) = 0$ is satisfied for every $x \in \mathcal{X}_j$ if and only if $F(\mathbf{a}_1, X_r) = 0$. We may express the latter by $d + 1$ linearly-independent linear equations on the coefficients of $F \in \mathcal{F}_{r,d}$. On the other hand, the equalities $F(\mathbf{a}_2, y) = 0$ for all $y \in \mathcal{Y}_k$ impose k additional linearly-independent conditions on the coefficients of F . We conclude that

$$\mathcal{N}_{j,k} = \sum_{\mathcal{X}_j, \mathcal{Y}_k \subset \mathbb{F}_q} q^{\dim \mathcal{F}_{r,d} - (d+1) - k} = \binom{q}{j} \binom{q}{k} q^{\dim \mathcal{F}_{r,d} - (d+1) - k}.$$

The third case, namely $j \leq d$ and $k > d$, is completely analogous to the second one. Finally, when $j > d$ and $k > d$, the conditions under consideration imply $F(\mathbf{a}_1, X_r) = F(\mathbf{a}_2, X_r) = 0$. We readily deduce that

$$\mathcal{N}_{j,k} = \binom{q}{j} \binom{q}{k} q^{\dim \mathcal{F}_{r,d} - 2d - 1}.$$

From the expression for $\mathcal{N}_{j,k}$ of the four cases under consideration we infer that

$$\begin{aligned} \frac{N_{\mathbf{a},2}}{|\mathcal{F}_{r,d}|} &= \frac{1}{|\mathcal{F}_{r,d}|} \sum_{j=1}^q \sum_{k=1}^q (-1)^{j+k} \mathcal{N}_{j,k} \\ &= \sum_{j=1}^d \sum_{k=1}^d (-1)^{j+k} \binom{q}{j} \binom{q}{k} q^{-j-k} + 2 \sum_{j=1}^d \sum_{k=d+1}^q (-1)^{j+k} \binom{q}{j} \binom{q}{k} q^{-j-(d+1)} \\ &\quad + \sum_{j=d+1}^q \sum_{k=d+1}^q (-1)^{j+k} \binom{q}{j} \binom{q}{k} q^{-2d-1}. \end{aligned}$$

By (2.4) and elementary calculations we obtain

$$\begin{aligned} \frac{N_{\mathbf{a},2}}{|\mathcal{F}_{r,d}|} &= \left(\sum_{j=1}^d (-1)^j \binom{q}{j} q^{-j} \right)^2 - 2 \left(\sum_{j=1}^d (-1)^j \binom{q}{j} q^{-j} \right) (-1)^d \binom{q-1}{d} q^{-d-1} \\ &\quad + \binom{q-1}{d}^2 q^{-2d-1}. \end{aligned}$$

This and Theorem 2.1 readily imply the claim. \square

Combining the previous claim and Theorem 2.1 we deduce that

$$\begin{aligned} p_{r,d}[C_{\mathbf{a}} = 2] &= P_1[C_1 = 1] - \frac{N_{\mathbf{a},2}}{|\mathcal{F}_{r,d}|} \\ &= (1 - P_1[C_1 = 1])P_1[C_1 = 1] - \frac{q-1}{q^{2d+2}} \binom{q-1}{d}^2. \end{aligned}$$

Let $g : \mathbb{R} \rightarrow \mathbb{R}$, $g(x) := (1-x)x$. The Mean Value theorem shows that there exists $\xi \in (0, 1)$ such that

$$(1 - P_1[C_1 = 1])P_1[C_1 = 1] - (1 - \mu_d)\mu_d = g'(\xi) (P_1[C_1 = 1] - \mu_d).$$

As the function $x \mapsto g'(x)$ maps the real interval $[0, 1]$ to $[-1, 1]$, we conclude that $|g'(\xi)| \leq 1$. Therefore, from Corollary 2.2 it follows that

$$|(1 - P_1[C_1 = 1])P_1[C_1 = 1] - (1 - \mu_d)\mu_d| \leq |P_1[C_1 = 1] - \mu_d| \leq \frac{2}{q}.$$

On the other hand, it is easy to see that $\frac{q-1}{q^{2d+2}} \binom{q-1}{d}^2 \leq 1/q$. This immediately implies the statement of the proposition. \square

Proposition 2.4 is the critical step in the analysis of the behavior of the probability $P_2[C_2 = 2]$, which is estimated in the next result.

Theorem 2.5. *For any $q > d$,*

$$|P_2[C_2 = 2] - (1 - \mu_d)\mu_d| \leq \frac{3}{q}.$$

Proof. By Lemma 2.3 and Proposition 2.4 we obtain

$$|P_2[C_2 = 2] - (1 - \mu_d)\mu_d| \leq \frac{1}{N_2} \sum_{\mathbf{a} \in \mathbb{F}_2} |p_{r,d}[C_{\mathbf{a}} = 2] - (1 - \mu_d)\mu_d| \leq \frac{3}{q}.$$

This finishes the proof of the theorem. \square

We finish the section with a remark concerning the spaces considered so far to discuss the probability that the SVS algorithm performs at most two searches on vertical strips. For the analysis of the probability of one search we have considered $\mathbf{F}_1 := \mathbb{F}_q^{r-1}$ and the random variable $C_1 : \mathbf{F}_1 \times \mathcal{F}_{r,d} \rightarrow \{1, \infty\}$, while in the analysis of the probability of two searches we have considered the random variable $C_2 : \mathbf{F}_2 \times \mathcal{F}_{r,d} \rightarrow \{1, 2, \infty\}$. To link both analyses, in Lemma 4.1 below we prove that

$$P_2[C_2 = 1] = P_1[C_1 = 1],$$

which shows the consistency of the probability spaces underlying Theorems 2.1 and 2.5. In Section 4 we shall show that the analysis of the probability that s vertical strips are searched can be done in a unified framework for any $s \geq 1$.

3. THE NUMBER OF SEARCHES FOR GIVEN VERTICAL STRIPS

As can be inferred from Section 2, a critical step in the probabilistic analysis of SVS algorithm is the determination of the probability of s searches, for a given choice of s vertical strips. The cases $s = 1$ and $s = 2$ were discussed in Section 2. In this section we carry out the analysis of the general case.

Fix $3 \leq s \leq \min\{\binom{d+r-1}{r-1}, q^{r-1}\}$ and $\mathbf{a}_1, \dots, \mathbf{a}_s \in \mathbb{F}_q^{r-1}$ with $\mathbf{a}_i \neq \mathbf{a}_j$ for $i \neq j$. Denote $\underline{\mathbf{a}} := (\mathbf{a}_1, \dots, \mathbf{a}_s)$. Assuming that $\underline{\mathbf{a}}$ is the choice for the first s vertical strips to be considered, we analyze the probability that the SVS algorithm finds an \mathbb{F}_q -rational zero of the polynomial under consideration in the s th search.

For this purpose, we consider the set $\mathcal{F}_{r,d}$ endowed with the uniform probability $p_{r,d}$ and the random variable $C_{\underline{\mathbf{a}}} := C_{\underline{\mathbf{a}},r,d} : \mathcal{F}_{r,d} \rightarrow \{1, 2, \dots, s, \infty\}$ which counts the number of searches for a given input on the vertical strips determined by $\mathbf{a}_1, \dots, \mathbf{a}_s$, $C_{\underline{\mathbf{a}}}(F) = \infty$ meaning that F has no \mathbb{F}_q -rational zeros on these vertical strips.

We start with the following elementary result.

Lemma 3.1. *Let \mathbb{V} and \mathbb{W} be \mathbb{F}_q -linear spaces of finite dimension and $\Phi : \mathbb{V} \rightarrow \mathbb{W}$ any \mathbb{F}_q -linear mapping. Consider \mathbb{V} and \mathbb{W} endowed with the uniform probabilities $P_{\mathbb{V}}$ and $P_{\mathbb{W}}$ respectively. Then for any $A \subset \mathbb{W}$ we have*

$$P_{\mathbb{V}}(\Phi^{-1}(A)) = \frac{|A \cap \text{Im}(\Phi)|}{|\text{Im}(\Phi)|} = \frac{P_{\mathbb{W}}(A \cap \text{Im}(\Phi))}{P_{\mathbb{W}}(\text{Im}(\Phi))} =: P_{\text{Im}(\Phi)}(A).$$

Proof. We have

$$\frac{1}{|\mathbb{V}|} |\Phi^{-1}(A)| = \frac{1}{|\mathbb{V}|} \sum_{\mathbf{w} \in A} |\Phi^{-1}(\mathbf{w})| = \frac{1}{|\mathbb{V}|} |\text{Ker}(\Phi)| |A \cap \text{Im}(\Phi)|.$$

By the Dimension theorem and the equality $|\mathbb{S}| = q^{\dim \mathbb{S}}$, valid for any \mathbb{F}_q -vector space \mathbb{S} , we see that $|\mathbb{V}| = |\text{Ker}(\Phi)| |\text{Im}(\Phi)|$. Then

$$\frac{1}{|\mathbb{V}|} |\Phi^{-1}(A)| = \frac{|A \cap \text{Im}(\Phi)|}{|\text{Im}(\Phi)|} = \frac{P_{\mathbb{W}}(A \cap \text{Im}(\Phi))}{P_{\mathbb{W}}(\text{Im}(\Phi))}.$$

This finishes the proof of the lemma. \square

For simplicity of notations, we replace the variable X_r by a new indeterminate T and consider the \mathbb{F}_q -linear mapping $\Phi := \Phi_{\underline{\mathbf{a}}} : \mathcal{F}_{r,d} \rightarrow \mathcal{F}_{1,d}^s$ defined as

$$(3.1) \quad \Phi(F) := (F(\mathbf{a}_1, T), \dots, F(\mathbf{a}_s, T)).$$

Since $\text{Im}(\Phi)$ is an \mathbb{F}_q -linear space, by Lemma 3.1 it follows that

$$(3.2) \quad p_{r,d}[C_{\underline{\mathbf{a}}} = s] = \frac{|(\{N = 0\}^{s-1} \times \{N > 0\}) \cap \text{Im}(\Phi)|}{|\text{Im}(\Phi)|},$$

where $N := N_{1,d}$ denotes the random variable which counts the number of zeros in \mathbb{F}_q of the elements of $\mathcal{F}_{1,d}$. As a consequence, we need to estimate the quantity

$$R_s := |(\{N = 0\}^{s-1} \times \{N > 0\}) \cap \text{Im}(\Phi)|.$$

In the next section we obtain a characterization of the image of Φ that will allow us to express R_s in terms of the average cardinality of the value set of certain families of univariate polynomials. This is the critical step to estimate the quantity R_s .

As we explain below, there exists a unique positive integer $\kappa_s \leq d$ such that

$$\binom{\kappa_s + r - 2}{r - 1} < s \leq \binom{\kappa_s + r - 1}{r - 1}.$$

In the sequel we shall assume that the points $\mathbf{a}_1, \dots, \mathbf{a}_s$ under consideration satisfy the condition we now state. For $1 \leq j \leq \kappa_s$, let $D_j := \binom{j+r-1}{r-1}$ and denote by $\Omega_j := \{\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_{D_j}\} \subset (\mathbb{Z}_{\geq 0})^{r-1}$ the set of $(r-1)$ -tuples $\boldsymbol{\omega}_k := (\omega_{k,1}, \dots, \omega_{k,r-1})$ with $|\boldsymbol{\omega}_k| := \omega_{k,1} + \dots + \omega_{k,r-1} \leq j$. Let $\mathbf{a}_i^{\boldsymbol{\omega}_k} := a_{i,1}^{\omega_{k,1}} \dots a_{i,r-1}^{\omega_{k,r-1}}$ for $1 \leq i \leq s$ and $1 \leq k \leq D_j$. Then we require that the multivariate Vandermonde matrix

$$(3.3) \quad \mathcal{M}_j := \begin{pmatrix} \mathbf{a}_1^{\boldsymbol{\omega}_1} & \dots & \mathbf{a}_1^{\boldsymbol{\omega}_{D_j}} \\ \vdots & & \vdots \\ \mathbf{a}_s^{\boldsymbol{\omega}_1} & \dots & \mathbf{a}_s^{\boldsymbol{\omega}_{D_j}} \end{pmatrix} \in \mathbb{F}_q^{s \times D_j}$$

has maximal rank $\min\{D_j, s\}$ for $1 \leq j \leq \kappa_s$.

We briefly argue that this is a mild requirement which is likely to be satisfied by any “reasonable” choice of the elements $\mathbf{a}_1, \dots, \mathbf{a}_s \in \mathbb{F}_q^{r-1}$. Let $\mathbf{A}_1, \dots, \mathbf{A}_s$ be $(r-1)$ -tuples of indeterminates over $\overline{\mathbb{F}}_q$, that is, $\mathbf{A}_i := (A_{i,1}, \dots, A_{i,r-1})$ for $1 \leq i \leq s$, and denote by \mathcal{V}_j the following $\min\{D_j, s\} \times \min\{D_j, s\}$ Vandermonde matrix with entries in $\mathbb{F}_q[\mathbf{A}_1, \dots, \mathbf{A}_s]$:

$$\mathcal{V}_j := \begin{pmatrix} \mathbf{A}_1^{\boldsymbol{\omega}_1} & \dots & \mathbf{A}_1^{\boldsymbol{\omega}_{\min\{D_j, s\}}} \\ \vdots & & \vdots \\ \mathbf{A}_{\min\{D_j, s\}}^{\boldsymbol{\omega}_1} & \dots & \mathbf{A}_{\min\{D_j, s\}}^{\boldsymbol{\omega}_{\min\{D_j, s\}}} \end{pmatrix}.$$

Assume that the numbering of $\Omega_j := \{\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_{D_j}\} \subset (\mathbb{Z}_{\geq 0})^{r-1}$ is made according to degrees, i.e., $|\boldsymbol{\omega}_k| \leq |\boldsymbol{\omega}_l|$ whenever $k \leq l$. In particular, $\boldsymbol{\omega}_1 = (0, \dots, 0)$. By [10, Theorem 1.5] it follows that $\det \mathcal{V}_j$ is absolutely irreducible, namely it is a nonzero irreducible element of $\overline{\mathbb{F}}_q[\mathbf{A}_1, \dots, \mathbf{A}_s]$, for $1 \leq j \leq \kappa_s$. Let δ_j denote the degree of $\det \mathcal{V}_j$. We have the bound $\delta_j \leq jD_j$. Then [6, Theorem 5.2] proves that the number \mathcal{N}_j of $(r-1)$ -tuples $\mathbf{a}_1, \dots, \mathbf{a}_s \in \mathbb{F}_q^{r-1}$ annihilating $\det \mathcal{V}_j$ satisfies the estimate

$$(3.4) \quad |\mathcal{N}_j - q^{s(r-1)-1}| \leq (\delta_j - 1)(\delta_j - 2)q^{s(r-1)-\frac{3}{2}} + 5\delta_j^{\frac{13}{3}}q^{s(r-1)-2}.$$

Any choice of $\mathbf{a}_1, \dots, \mathbf{a}_s$ avoiding these $\mathcal{N}_j = \mathcal{O}(q^{s(r-1)-1})$ tuples for $1 \leq j \leq \kappa_s$ will satisfy our requirements. Furthermore, many “bad” choices $\mathbf{a}_1, \dots, \mathbf{a}_s$ annihilating the polynomial $\det \mathcal{V}_j$ for a given j will also work, as other minors of the Vandermonde matrix \mathcal{M}_j of (3.3) may be nonsingular. In particular, for $s \leq r$ and $\mathbf{a}_1, \dots, \mathbf{a}_s$ affinely independent, our requirement is satisfied.

Summarizing, denote $\mathcal{V}^s := \prod_{j=1}^{\kappa_s} \det \mathcal{V}_j \in \mathbb{F}_q[\mathbf{A}_1, \dots, \mathbf{A}_s]$ and let

$$(3.5) \quad \mathbf{B}_s := \{\underline{\mathbf{a}} := (\mathbf{a}_1, \dots, \mathbf{a}_s) \in \mathbb{F}_q^{s(r-1)} : \mathcal{V}^s(\underline{\mathbf{a}}) = 0\}.$$

Then $|\mathbf{B}_s| = \mathcal{O}(q^{s(r-1)-1})$ and all the results of this section are valid for any $\underline{\mathbf{a}} \in \mathbb{F}_q^{s(r-1)} \setminus \mathbf{B}_s$.

3.1. A characterization of the image of Φ . In order to characterize the image $\text{Im}(\Phi)$, we shall express each element of $\mathcal{F}_{r,d}$ by its coordinates in the standard monomial basis \mathcal{B} of $\mathcal{F}_{r,d}$, considering the monomial order we now define. Denote by \mathcal{B}_i the set of monomials of $\mathbb{F}_q[X_1, \dots, X_{r-1}]$ of degree at most i for $0 \leq i \leq d$, with the standard lexicographical order defined by setting $X_1 < X_2 < \dots < X_{r-1}$. The basis \mathcal{B} is considered with the order $\mathcal{B} = \{X_r^d, X_r^{d-1}\mathcal{B}_1, \dots, X_r\mathcal{B}_{d-1}, \mathcal{B}_d\}$, where each set $X_r^{d-i}\mathcal{B}_i$ is ordered following the order induced by the one of \mathcal{B}_i . In other words, any $F \in \mathcal{F}_{r,d}$ can be uniquely expressed as

$$F = \sum_{i=0}^d F_i(X_1, \dots, X_{r-1})X_r^i,$$

where each F_i has degree at most $d-i$ for $0 \leq i \leq d$. Then the vector of coefficients $(F)_{\mathcal{B}}$ of F in the basis \mathcal{B} is given by $(F)_{\mathcal{B}} = ((F_d)_{\mathcal{B}_0}, \dots, (F_0)_{\mathcal{B}_d})$. On the other hand, we shall express the elements of $\mathcal{F}_{1,d}^s$ in the basis $\mathcal{B}' := \{T^d, \dots, T, 1\}^s$.

Let

$$D_j := \binom{j+r-1}{r-1} = |\mathcal{B}_j| \quad (0 \leq j \leq d), \quad D := \binom{d+r}{r} = |\mathcal{B}| = \sum_{j=0}^d |\mathcal{B}_j|.$$

We also set $D_{-1} := 0$. Observe that the sequence $(D_j)_{j \geq -1}$ is strictly increasing. Therefore, for each i with $1 \leq i \leq s$ there exists a unique $\kappa_i \in \mathbb{N}$ such that

$$(3.6) \quad D_{\kappa_i-1} < i \leq D_{\kappa_i}.$$

The following remarks can be easily established.

Remark 3.2.

- $\kappa_i \leq j$ if and only if $i \leq D_j$.
- $\kappa_1 = 0$, $\kappa_s \leq d$.

The matrix $\mathbf{M}_{\Phi} \in \mathbb{F}_q^{s(d+1) \times D}$ of Φ with respect to the bases defined above can be written as the following block matrix:

$$\mathbf{M}_{\Phi} = \begin{pmatrix} \mathbf{M}_1 \\ \vdots \\ \mathbf{M}_s \end{pmatrix},$$

where $\mathbf{M}_i \in \mathbb{F}_q^{(d+1) \times D}$ is the diagonal block matrix

$$\mathbf{M}_i := \begin{pmatrix} \mathbf{M}_{i,0} & & & \\ & \mathbf{M}_{i,1} & & \\ & & \ddots & \\ & & & \mathbf{M}_{i,d} \end{pmatrix}, \quad \mathbf{M}_{i,j} := (\mathbf{a}_i^{\alpha} : |\alpha| \leq j) \in \mathbb{F}_q^{1 \times D_j}.$$

Our first result concerns the dimension of $\text{Im}(\Phi)$.

Lemma 3.3. *For $s \leq \min\{D_d, q^{r-1}\}$, we have*

$$\dim \text{Im}(\Phi) = \binom{\kappa_s - 1 + r}{r} + s(d - \kappa_s + 1) = \sum_{i=1}^s (d + 1 - \kappa_i).$$

Proof. Let $\mathbf{h} := (h_1, \dots, h_s)$ be an element of $\text{Im}(\Phi)$. Then there exists $F \in \mathcal{F}_{r,d}$ with $\mathbf{h} = \Phi(F)$. Denote by $(F)_{\mathcal{B}} = ((F_d)_{\mathcal{B}_0}, \dots, (F_0)_{\mathcal{B}_d})$ the coordinates of F in the basis \mathcal{B} . Then the block structure of the matrix \mathbf{M}_{Φ} implies

$$(3.7) \quad \Phi(F) = \sum_{j=0}^d \begin{pmatrix} \mathbf{M}_{1,j} \\ \vdots \\ \mathbf{M}_{s,j} \end{pmatrix} (F_{d-j})_{\mathcal{B}_j} T^{d-j}.$$

As $\underline{a} \notin \mathcal{B}_s$, we have

$$\text{rank} \begin{pmatrix} \mathbf{M}_{1,j} \\ \vdots \\ \mathbf{M}_{s,j} \end{pmatrix} = \min\{D_j, s\} = \begin{cases} D_j & \text{for } 0 \leq j \leq \kappa_s - 1, \\ s & \text{for } \kappa_s \leq j \leq d. \end{cases}$$

As a consequence,

$$\dim \text{Im}(\Phi) = \sum_{j=0}^{\kappa_s-1} D_j + s(d - \kappa_s + 1) = \binom{\kappa_s - 1 + r}{r} + s(d - \kappa_s + 1).$$

This proves the first assertion of the lemma. To prove the second assertion, we have

$$\begin{aligned} \sum_{i=1}^s (d+1-\kappa_i) &= \sum_{j=0}^{\kappa_s} \sum_{i=D_{j-1}+1}^{\min\{D_j, s\}} (d+1-j) \\ &= \sum_{j=0}^{\kappa_s-1} (d+1-j)(D_j - D_{j-1}) + (d+1-\kappa_s)(s - D_{\kappa_s-1}). \end{aligned}$$

Since $\sum_{j=0}^k (D_j - D_{j-1}) = D_k$, we conclude that

$$\sum_{i=1}^s (d+1-\kappa_i) = - \sum_{j=0}^{\kappa_s-1} j(D_j - D_{j-1}) + (d+1-\kappa_s)s + \kappa_s D_{\kappa_s-1}.$$

Taking into account the identity $\sum_{j=0}^K j \binom{j+R}{R} = (R+1) \binom{R+1+K}{R+2}$, we obtain

$$\sum_{i=1}^s (d+1-\kappa_i) = -(r-1) \binom{\kappa_s + r - 2}{r} + (d+1-\kappa_s)s + \kappa_s D_{\kappa_s-1}.$$

A simple calculation finishes the proof of the lemma. \square

Next we determine a suitable parameterization of $\text{Im}(\Phi)$. To this end, let $\Phi^* : \text{Im}(\Phi) \rightarrow \mathbb{F}_q^{\dim \text{Im}(\Phi)}$ be the \mathbb{F}_q -linear mapping defined by

$$\Phi^*(\mathbf{h}) := \mathbf{h}^*,$$

where $\mathbf{h} := (h_1, \dots, h_s)$, $h_i := (h_{d,i}, \dots, h_{0,i}) \in \mathbb{F}_q^{d+1}$ for $1 \leq i \leq s$ and

$$(3.8) \quad \mathbf{h}^* := (h_1^*, \dots, h_s^*), \quad h_i^* := (h_{d-\kappa_i,i}, \dots, h_{0,i}) \quad (1 \leq i \leq s).$$

Lemma 3.3 shows that Φ^* is well-defined.

Lemma 3.4. Φ^* is an isomorphism.

Proof. Since Φ^* is a linear mapping between \mathbb{F}_q -vector spaces of the same dimension, it suffices to show that Φ^* is injective. Fix $\mathbf{h} := \Phi(F) \in \text{Im}(\Phi)$ with $\mathbf{h}^* = \mathbf{0}$. From (3.7) we deduce that

$$(3.9) \quad \begin{pmatrix} M_{1,j} \\ \vdots \\ M_{s,j} \end{pmatrix} (F_{d-j})_{\mathcal{B}_j} = \begin{pmatrix} h_{d-j,1} \\ \vdots \\ h_{d-j,s} \end{pmatrix}.$$

Fix j with $0 \leq j \leq \kappa_s - 1$. Then the element $h_{d-j,i}$ is included in the definition of h_i^* if and only if $i \leq D_j$ (see Remark 3.2). As $\mathbf{h}^* = \mathbf{0}$ by hypothesis, it follows that $h_{d-j,i} = 0$ for $1 \leq i \leq D_j$ and we have the identity

$$\begin{pmatrix} M_{1,j} \\ \vdots \\ M_{D_j,j} \\ M_{D_j+1,j} \\ \vdots \\ M_{s,j} \end{pmatrix} (F_{d-j})_{\mathcal{B}_j} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ h_{d-j,D_j+1} \\ \vdots \\ h_{d-j,s} \end{pmatrix}.$$

Since the upper $(D_j \times D_j)$ -submatrix of the matrix in the left-hand side is invertible, we conclude that $(F_{d-j})_{\mathcal{B}_j} = \mathbf{0}$. This implies $h_{d-j,D_j+1} = \dots = h_{d-j,s} = 0$. On the other hand, for $j \geq \kappa_s$ the element $h_{d-j,i}$ is included in the definition of h_i^* for $1 \leq i \leq s$ and therefore $h_{d-j,i} = 0$ for $1 \leq i \leq s$. This shows that $\mathbf{h} = \mathbf{0}$. \square

Denote by $\Psi := (\psi_1, \dots, \psi_s) : \mathbb{F}_q^{\dim \text{Im}(\Phi)} \rightarrow \text{Im}(\Phi)$ the inverse mapping of Φ^* . We need further information concerning the mappings ψ_i .

Lemma 3.5. *Let be given $h_i^* := (h_{d-\kappa_i,i}, \dots, h_{0,i}) \in \mathbb{F}_q^{d+1-\kappa_i}$ for $1 \leq i \leq s$. Let $\mathbf{h}^* := (h_1^*, \dots, h_s^*) \in \mathbb{F}_q^{\dim \text{Im}(\Phi)}$ and $\mathbf{h} := \Psi(\mathbf{h}^*)$. Denote*

$$h_i := \psi_i(\mathbf{h}^*) := h_{d,i}T^d + \dots + h_{d+1-\kappa_i,i}T^{d+1-\kappa_i} + h_{d-\kappa_i,i}T^{d-\kappa_i} + \dots + h_{0,i}.$$

Then $h_{d,i}, \dots, h_{d+1-\kappa_i,i}$ are uniquely determined by h_1^, \dots, h_{i-1}^* .*

Proof. Fix k with $0 \leq k \leq \kappa_i - 1$. Write $\mathbf{h} := \Phi(F)$. In the proof of Lemma 3.3 we prove that

$$\begin{pmatrix} M_{1,k} \\ \vdots \\ M_{D_k,k} \end{pmatrix} (F_{d-k})_{\mathcal{B}_k} = \begin{pmatrix} h_{d-k,1} \\ \vdots \\ h_{d-k,D_k} \end{pmatrix},$$

where the $(D_k \times D_k)$ -matrix in the left-hand side is invertible. The element $h_{d-k,l}$ is included in the definition of h_l^* if and only if $l \leq D_k$. Furthermore, we have $k \leq \kappa_i - 1 \leq \kappa_{i-1}$. We conclude that the vector in the right-hand side is uniquely determined by h_1^*, \dots, h_{i-1}^* , and thus so is $(F_{d-k})_{\mathcal{B}_k}$. Therefore, the identity

$$\begin{pmatrix} M_{1,k} \\ \vdots \\ M_{i,k} \end{pmatrix} (F_{d-k})_{\mathcal{B}_k} = \begin{pmatrix} h_{d-k,1} \\ \vdots \\ h_{d-k,i} \end{pmatrix}$$

shows that the element $h_{d-k,i}$ is uniquely determined by h_1^*, \dots, h_{i-1}^* . \square

We end this section with the following remark.

Remark 3.6. For each $\mathbf{h} := (h_1, \dots, h_s) \in \text{Im}(\Phi)$, we have $h_{d,1} = \dots = h_{d,s}$. Indeed, from (3.7) we deduce that

$$\begin{pmatrix} M_{1,0} \\ \vdots \\ M_{s,0} \end{pmatrix} (F_d)_{\mathcal{B}_0} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} (F_d)_{\mathcal{B}_0} = \begin{pmatrix} h_{d,1} \\ \vdots \\ h_{d,s} \end{pmatrix}.$$

This implies $h_{d,1} = \dots = h_{d,s} = (F_d)_{\mathcal{B}_0}$. In particular, the coefficient $h_{d,1}$ of the monomial T^d in the polynomial h_1 uniquely determines the coefficient $h_{d,j}$ of the monomial T^d in h_j for $2 \leq j \leq s$. \square

3.2. The probability of s searches in terms of cardinalities of value sets.

For $\underline{a} := (a_1, \dots, a_s) \in \mathbb{F}_q^{s(r-1)} \setminus \mathcal{B}_s$ as before, we need to estimate the quantity

$$R_s := |(\{N=0\}^{s-1} \times \{N>0\}) \cap \text{Im}(\Phi)|.$$

According to Lemma 3.4, each element $\mathbf{h} \in \text{Im}(\Phi)$ can be uniquely expressed in the form $\mathbf{h} = \Psi(\mathbf{h}^*)$, where \mathbf{h}^* is defined as in (3.8). Hence,

$$(3.10) \quad R_s = \sum_{\mathbf{h}^* \in \mathbb{F}_q^{\dim \text{Im}(\Phi)}} \mathbf{1}_{\{N=0\}^{s-1} \times \{N>0\}}(\Psi(\mathbf{h}^*)),$$

where $\mathbf{1}_{\{N=0\}^{s-1} \times \{N>0\}} : \mathcal{F}_{1,d}^s \rightarrow \{0,1\}$ denotes the characteristic function of the set $\{N=0\}^{s-1} \times \{N>0\}$. By Lemma 3.5, the coordinate $\psi_i(\mathbf{h}^*)$ depends only on $\mathbf{h}_i^* := (h_1^*, \dots, h_i^*)$ for $1 \leq i \leq s$. We shall therefore write $\psi_i(\mathbf{h}^*)$ as $\psi_i(\mathbf{h}_i^*)$ for $1 \leq i \leq s$, with a slight abuse of notation.

First, we rewrite the expression (3.10) for R_s in a suitable form for our purposes.

Lemma 3.7. *Let $\mathbf{h} := (\sum_{j=0}^d h_{j,1}T^j, \dots, \sum_{j=0}^d h_{j,s}T^j)$ be an arbitrary element of $\text{Im}(\Phi)$ and let $\mathbf{h}^* := \Phi^*(\mathbf{h}) := (h_1^*, \dots, h_s^*) \in \mathbb{F}_q^{\dim \text{Im}(\Phi)}$ be defined as in (3.8). For $s \leq \min\{D_d, q^{r-1}\}$, the following identity holds:*

$$R_s = \sum_{\substack{h_1^* \in \mathbb{F}_q^{d+1} \\ N(\psi_1(\mathbf{h}_1^*))=0}} \dots \sum_{\substack{h_{s-1}^* \in \mathbb{F}_q^{d+1-\kappa_{s-1}} \\ N(\psi_{s-1}(\mathbf{h}_{s-1}^*))=0}} \sum_{h_s^* \in \mathbb{F}_q^{d+1-\kappa_s}} \mathbf{1}_{\{N>0\}}(\psi_s(\mathbf{h}_s^*)).$$

Proof. We may rewrite (3.10) in the following way:

$$R_s = \sum_{h_1^* \in \mathbb{F}_q^{d+1}} \cdots \sum_{h_s^* \in \mathbb{F}_q^{d+1-\kappa_s}} \mathbf{1}_{\{N=0\}^{s-1} \times \{N>0\}}(\Psi(\mathbf{h}^*)).$$

As a consequence of the remarks before the statement of Lemma 3.7, it follows that

$$\begin{aligned} \mathbf{1}_{\{N=0\}^{s-1} \times \{N>0\}}(\Psi(\mathbf{h}^*)) &= \prod_{i=1}^{s-1} \mathbf{1}_{\{N=0\}}(\psi_i(\mathbf{h}^*)) \cdot \mathbf{1}_{\{N>0\}}(\psi_s(\mathbf{h}^*)) \\ &= \prod_{i=1}^{s-1} \mathbf{1}_{\{N=0\}}(\psi_i(\mathbf{h}_i^*)) \cdot \mathbf{1}_{\{N>0\}}(\psi_s(\mathbf{h}_s^*)). \end{aligned}$$

Then the previous expression for R_s can be rewritten as follows:

$$R_s = \sum_{h_1^* \in \mathbb{F}_q^{d+1}} \mathbf{1}_{\{N=0\}}(\psi_1(\mathbf{h}_1^*)) \cdots \sum_{h_{s-1}^* \in \mathbb{F}_q^{d+1-\kappa_{s-1}}} \mathbf{1}_{\{N=0\}}(\psi_{s-1}(\mathbf{h}_{s-1}^*)) \sum_{h_s^* \in \mathbb{F}_q^{d+1-\kappa_s}} \mathbf{1}_{\{N>0\}}(\psi_s(\mathbf{h}_s^*)),$$

which readily implies the lemma. \square

For $1 \leq i \leq s-1$, fix $h_i^* \in \mathbb{F}_q^{d+1-\kappa_i}$. For each $h_s^* := (h_{d-\kappa_s, s}, \dots, h_{0, s}) \in \mathbb{F}_q^{d+1-\kappa_s}$, denote by $f_{h_s^*}$ the polynomial

$$f_{h_s^*} := \psi_s(h_1^*, \dots, h_s^*) := h_{d, s} T^d + \cdots + h_{d+1-\kappa_s, s} T^{d+1-\kappa_s} + h_{d-\kappa_s, s} T^{d-\kappa_s} + \cdots + h_{0, s}.$$

According to Lemma 3.7, we are interested in estimating the sum

$$(3.11) \quad \sum_{h_s^* \in \mathbb{F}_q^{d+1-\kappa_s}} \mathbf{1}_{\{N>0\}}(f_{h_s^*}).$$

For $h_s^* := (h_{d-\kappa_s, s}, \dots, h_{0, s}) \in \mathbb{F}_q^{d+1-\kappa_s}$, denote $\hat{h}_s^* := (h_{d-\kappa_s, s}, \dots, h_{1, s}) \in \mathbb{F}_q^{d-\kappa_s}$ and $f_{\hat{h}_s^*} := \sum_{j=1}^d h_{j, s} T^j = f_{h_s^*} - f_{h_s^*}(0)$. We observe that

$$\begin{aligned} \sum_{h_s^* \in \mathbb{F}_q^{d+1-\kappa_s}} \mathbf{1}_{\{N>0\}}(f_{h_s^*}) &= \sum_{\hat{h}_s^* \in \mathbb{F}_q^{d-\kappa_s}} \sum_{h_{0, s} \in \mathbb{F}_q} \mathbf{1}_{\{N>0\}}(f_{h_s^*}) = \sum_{\hat{h}_s^* \in \mathbb{F}_q^{d-\kappa_s}} \mathcal{V}(f_{\hat{h}_s^*}) \\ (3.12) \quad &= \frac{1}{q} \sum_{h_s^* \in \mathbb{F}_q^{d+1-\kappa_s}} \mathcal{V}(f_{h_s^*}), \end{aligned}$$

where $\mathcal{V}(f) := |\{f(c) : c \in \mathbb{F}_q\}|$ is the cardinality of the value set of $f \in \mathbb{F}_q[T]$. Lemma 3.5 proves that $h_{d, s}, \dots, h_{d+1-\kappa_s, s}$ are uniquely determined by $\mathbf{h}_{s-1}^* := (h_1^*, \dots, h_{s-1}^*)$. Thus, the sum in the right-hand side of (3.12) takes as argument the cardinality of the value set of all the elements of $\mathcal{F}_{1, d}$ having its first κ_s coefficients $(h_{d, s}, \dots, h_{d+1-\kappa_s, s})$ prescribed. Set $\psi_s^{\text{fix}}(\mathbf{h}_{s-1}^*) := (h_{d, s}, \dots, h_{d+1-\kappa_s, s})$ and denote

$$(3.13) \quad \mathcal{V}_d(\kappa_s, \psi_s^{\text{fix}}(\mathbf{h}_{s-1}^*)) := \frac{1}{q^{d+1-\kappa_s}} \sum_{h_s^* \in \mathbb{F}_q^{d+1-\kappa_s}} \mathcal{V}(f_{h_s^*}).$$

Now we express the probability that $C_{\underline{a}} = s$ in terms of $\mathcal{V}_d(\kappa_s, \psi_s^{\text{fix}}(\mathbf{h}_{s-1}^*))$.

Lemma 3.8. *For $s \leq \min\{D_d, q^{r-1}\}$, the following identity holds:*

$$p_{r, d}[C_{\underline{a}} = s] = \frac{1}{q^{\sum_{i=1}^{s-1} (d+1-\kappa_i)}} \sum_{\substack{h_1^* \in \mathbb{F}_q^{d+1} \\ N(\psi_1(\mathbf{h}_1^*))=0}} \cdots \sum_{\substack{h_{s-1}^* \in \mathbb{F}_q^{d+1-\kappa_{s-1}} \\ N(\psi_{s-1}(\mathbf{h}_{s-1}^*))=0}} \frac{\mathcal{V}_d(\kappa_s, \psi_s^{\text{fix}}(\mathbf{h}_{s-1}^*))}{q}.$$

Proof. By Lemma 3.3 we know that $\dim \text{Im}(\Phi) = \sum_{i=1}^s (d+1-\kappa_i)$. Combining this with (3.2) and Lemma 3.7 we obtain

$$\begin{aligned} p_{r, d}[C_{\underline{a}} = s] &= \frac{1}{q^{\sum_{i=1}^{s-1} (d+1-\kappa_i)}} \sum_{\substack{h_1^* \in \mathbb{F}_q^{d+1} \\ N(\psi_1(\mathbf{h}_1^*))=0}} \cdots \sum_{\substack{h_{s-1}^* \in \mathbb{F}_q^{d+1-\kappa_{s-1}} \\ N(\psi_{s-1}(\mathbf{h}_{s-1}^*))=0}} \frac{1}{q^{d+1-\kappa_s}} \sum_{h_s^* \in \mathbb{F}_q^{d+1-\kappa_s+1}} \mathbf{1}_{\{N>0\}}(\psi_s(\mathbf{h}_s^*)). \end{aligned}$$

Then (3.12) and (3.13) complete the proof of the lemma. \square

If $s \leq \min\{D_{d-2}, q^{r-1}\}$, then, as we explain in the next section, for any \mathbf{h}_{s-1}^* such that $f_{h_s^*}$ is of degree d , the average cardinality in (3.13) has the asymptotic behavior $\mathcal{V}_d(\kappa_s, \psi_s^{\text{fix}}(\mathbf{h}_{s-1}^*)) = \mu_d q + \mathcal{O}(q^{1/2})$. Combining this with Lemma 3.8 we shall be led to consider “inner” sums in the expression for $p_{r,d}[C_{\underline{a}} = s]$, which shall be expressed in terms of the average cardinality of the value sets of the families of polynomials we now introduce. For $1 \leq i \leq s-1$ and $1 \leq j \leq i-1$, fix $h_j^* := (h_{d-\kappa_j, j}, \dots, h_{0, j}) \in \mathbb{F}_q^{d+1-\kappa_j}$. For each $h_i^* := (h_{d-\kappa_i, i}, \dots, h_{0, i}) \in \mathbb{F}_q^{d+1-\kappa_i}$, denote

$$f_{h_i^*} := \psi_i(h_1^*, \dots, h_i^*) := h_{d,i} T^d + \dots + h_{d+1-\kappa_i, i} T^{d+1-\kappa_i} + h_{d-\kappa_i, i} T^{d-\kappa_i} + \dots + h_{0, i}.$$

Lemma 3.5 proves that the coefficients $h_{d,i}, \dots, h_{d-\kappa_i+1, i}$ are uniquely determined by $\mathbf{h}_{i-1}^* := (h_1^*, \dots, h_{i-1}^*)$. Consequently, we set $\psi_i^{\text{fix}}(\mathbf{h}_{i-1}^*) := (h_{d,i}, \dots, h_{d+1-\kappa_i, i})$ and consider the average cardinality

$$(3.14) \quad \mathcal{V}_d(\kappa_i, \psi_i^{\text{fix}}(\mathbf{h}_{i-1}^*)) := \frac{1}{q^{d+1-\kappa_i}} \sum_{h_i^* \in \mathbb{F}_q^{d+1-\kappa_i}} \mathcal{V}(f_{h_i^*}).$$

Our next result expresses the probability of s searches in terms of the quantities $\mathcal{V}_d(\kappa_i, \psi_i^{\text{fix}}(\mathbf{h}_{i-1}^*))$ ($1 \leq i \leq s$).

Theorem 3.9. *For $s \leq \min\{D_d, q^{r-1}\}$, we have*

$$p_{r,d}[C_{\underline{a}} = s] = (1 - \mu_d)^{s-1} \mu_d \frac{q-1}{q} + \sum_{i=0}^s \mathcal{T}_i,$$

where $|\mathcal{T}_0| \leq 1/q$,

$$\mathcal{T}_i := (1 - \mu_d)^{s-i-1} \mu_d \frac{q-1}{q^{\sum_{j=1}^{i-1} (d+1-\kappa_j)}} \sum_{\substack{h_1^* \in \mathbb{F}_q^{d+1} \\ N(\psi_1(\mathbf{h}_1^*))=0 \\ h_{d,1}=1}} \dots \sum_{\substack{h_{i-1}^* \in \mathbb{F}_q^{d+1-\kappa_{i-1}} \\ N(\psi_{i-1}(\mathbf{h}_{i-1}^*))=0}} \left(\mu_d - \frac{\mathcal{V}_d(\kappa_i, \psi_i^{\text{fix}}(\mathbf{h}_{i-1}^*))}{q} \right)$$

for $1 \leq i \leq s-1$, and

$$\mathcal{T}_s := \frac{q-1}{q^{\sum_{i=1}^{s-1} (d+1-\kappa_i)}} \sum_{\substack{h_1^* \in \mathbb{F}_q^{d+1} \\ N(\psi_1(\mathbf{h}_1^*))=0 \\ h_{d,1}=1}} \dots \sum_{\substack{h_{s-1}^* \in \mathbb{F}_q^{d+1-\kappa_{s-1}} \\ N(\psi_{s-1}(\mathbf{h}_{s-1}^*))=0}} \left(\frac{\mathcal{V}_d(\kappa_s, \psi_s^{\text{fix}}(\mathbf{h}_{s-1}^*))}{q} - \mu_d \right).$$

Proof. Denote $C := C_{\underline{a}}$. We split the expression for $p_{r,d}[C = s]$ of Lemma 3.8 into two sums, depending on whether $h_{d,1} = 0$ or not. More precisely, we write

$$p_{r,d}[C = s] = p_{r,d}[C = s, F_d = 0] + p_{r,d}[C = s, F_d \neq 0],$$

where

$$\begin{aligned} p_{r,d}[C = s, F_d = 0] &= \frac{1}{q^{\sum_{i=1}^{s-1} (d+1-\kappa_i)}} \sum_{\substack{h_1^* \in \mathbb{F}_q^{d+1} \\ N(\psi_1(\mathbf{h}_1^*))=0 \\ h_{d,1}=0}} \dots \sum_{\substack{h_{s-1}^* \in \mathbb{F}_q^{d+1-\kappa_{s-1}} \\ N(\psi_{s-1}(\mathbf{h}_{s-1}^*))=0}} \frac{\mathcal{V}_d(\kappa_s, \psi_s^{\text{fix}}(\mathbf{h}_{s-1}^*))}{q}, \\ p_{r,d}[C = s, F_d \neq 0] &= \frac{1}{q^{\sum_{i=1}^{s-1} (d+1-\kappa_i)}} \sum_{\substack{h_1^* \in \mathbb{F}_q^{d+1} \\ N(\psi_1(\mathbf{h}_1^*))=0 \\ h_{d,1} \neq 0}} \dots \sum_{\substack{h_{s-1}^* \in \mathbb{F}_q^{d+1-\kappa_{s-1}} \\ N(\psi_{s-1}(\mathbf{h}_{s-1}^*))=0}} \frac{\mathcal{V}_d(\kappa_s, \psi_s^{\text{fix}}(\mathbf{h}_{s-1}^*))}{q}, \\ &= \frac{q-1}{q^{\sum_{i=1}^{s-1} (d+1-\kappa_i)}} \sum_{\substack{h_1^* \in \mathbb{F}_q^{d+1} \\ N(\psi_1(\mathbf{h}_1^*))=0 \\ h_{d,1}=1}} \dots \sum_{\substack{h_{s-1}^* \in \mathbb{F}_q^{d+1-\kappa_{s-1}} \\ N(\psi_{s-1}(\mathbf{h}_{s-1}^*))=0}} \frac{\mathcal{V}_d(\kappa_s, \psi_s^{\text{fix}}(\mathbf{h}_{s-1}^*))}{q}. \end{aligned}$$

In the first term we consider the intersection of the \mathbb{F}_q -linear space $\text{Im}(\Phi)$ with the linear subspace $\mathcal{F}_{1,d-1}^s$. As the former is not contained in the latter, the dimension of the intersection drops at least by one, and Lemma 3.1 implies

$$\mathcal{T}_0 := p_{r,d}[C = s, F_d = 0] \leq \frac{|\text{Im}(\Phi) \cap \mathcal{F}_{1,d-1}^s|}{|\text{Im}(\Phi)|} \leq \frac{q^{\dim \text{Im}(\Phi) - 1}}{q^{\dim \text{Im}(\Phi)}} = \frac{1}{q}.$$

On the other hand, it is easy to see that the expression for $p_{r,d}[C = s, F_d \neq 0]$ may be rewritten in the following way:

$$p_{r,d}[C = s, F_d \neq 0] = \mu_d \frac{q-1}{q^{\sum_{i=1}^{s-1} (d+1-\kappa_i)}} \sum_{\substack{h_1^* \in \mathbb{F}_q^{d+1} \\ N(\psi_1(h_1^*))=0 \\ h_{d,1}=1}} \cdots \sum_{\substack{h_{s-1}^* \in \mathbb{F}_q^{d+1-\kappa_{s-1}} \\ N(\psi_{s-1}(h_{s-1}^*))=0}} 1 + \mathcal{T}_s,$$

where \mathcal{T}_s is defined as in the statement of the theorem.

Now we claim that, for $1 \leq j \leq s$,

$$p_{r,d}[C = s, F_d \neq 0] = (1 - \mu_d)^{s-j} \mu_d \frac{q-1}{q^{\sum_{i=1}^{j-1} (d+1-\kappa_i)}} \sum_{\substack{h_1^* \in \mathbb{F}_q^{d+1} \\ N(\psi_1(h_1^*))=0 \\ h_{d,1}=1}} \cdots \sum_{\substack{h_{j-1}^* \in \mathbb{F}_q^{d+1-\kappa_{j-1}} \\ N(\psi_{j-1}(h_{j-1}^*))=0}} 1 + \sum_{i=j}^s \mathcal{T}_i,$$

where \mathcal{T}_i is defined as in the statement of the theorem. The claim for $j = 1$ is the assertion of the theorem.

We argue by downward induction on j from s to 1, the case $j = s$ being already proved. For $j < s$, suppose that the claim for $j+1$ is already established. We have

$$\frac{1}{q^{d+1-\kappa_j}} \sum_{\substack{h_j^* \in \mathbb{F}_q^{d+1-\kappa_j} \\ N(\psi_j(h_j^*))=0}} 1 = 1 - \frac{1}{q^{d+1-\kappa_j}} \sum_{\substack{h_j^* \in \mathbb{F}_q^{d+1-\kappa_j} \\ N(\psi_j(h_j^*))>0}} 1 = 1 - \frac{\mathcal{V}_d(\kappa_j, \psi_j^{\text{fix}}(h_{j-1}^*))}{q}.$$

Replacing this identity in the expression for $p_{r,d}[C = s, F_d \neq 0]$ corresponding to the claim for $j+1$ we readily deduce the claim for j , finishing thus the proof of the theorem. \square

3.3. The probability of $C_{\underline{a}} = s$. Theorem 3.9 shows that the probability that the SVS algorithm stops after $s \leq D_d$ attempts can be expressed in terms of the average cardinality $\mathcal{V}_d(\kappa_i, \psi_i^{\text{fix}}(h_{i-1}^*))$ of the value set of certain families of univariate polynomials for $1 \leq i \leq s$. Each of these families consists of all the polynomials

$$f_{\mathbf{b}} := \sum_{i=0}^{j-1} a_{d-i} T^{d-i} + \sum_{i=j}^d b_{d-i} T^{d-i}$$

with $\mathbf{b} := (b_{d-j}, \dots, b_0) \in \mathbb{F}_q^{d+1-j}$, for a given $1 \leq j \leq d$ and $\mathbf{a} := (a_d, \dots, a_{d-j}) \in \mathbb{F}_q^{j-1}$ with $a_d \neq 0$ (due to Remark 3.6). We are interested in the average

$$\mathcal{V}_d(j, \mathbf{a}) := \frac{1}{q^{d+1-j}} \sum_{\mathbf{b} \in \mathbb{F}_q^{d+1-j}} \mathcal{V}(f_{\mathbf{b}}).$$

Suppose that $q > d$. In [8], the following estimate is obtained for $1 \leq j \leq d/2 - 1$:

$$(3.15) \quad |\mathcal{V}_d(j, \mathbf{a}) - \mu_d q| \leq \frac{e^{-1}}{2} + \frac{(d-2)^5 e^{2\sqrt{d}}}{2^{d-2}} + \frac{7}{q}.$$

On the other hand, in [21] it is proved that, if the characteristic p of \mathbb{F}_q is greater than 2 and $1 \leq j \leq d-3$, then

$$(3.16) \quad |\mathcal{V}_d(j, \mathbf{a}) - \mu_d q| \leq d^2 2^{d-1} q^{\frac{1}{2}} + 133 d^{d+5} e^{2\sqrt{d}-d}.$$

Estimates (3.15) and (3.16) are the key point to determine the asymptotic behavior of the right-hand side of the expression for $p_{r,d}[C_{\underline{a}} = s]$ of Theorem 3.9. More precisely, we have the following result.

Theorem 3.10. *Let be given $\underline{a} := (\mathbf{a}_1, \dots, \mathbf{a}_s) \in \mathbb{F}_q^{s(r-1)} \setminus \mathbf{B}_s$, where the set \mathbf{B}_s is defined in (3.5). For $s \leq \min \left\{ \binom{d/2+r-1}{r-1}, q^{r-1} \right\}$, we have*

$$|p_{r,d}[C_{\underline{a}} = s] - (1 - \mu_d)^{s-1} \mu_d| \leq \left(e^{-1} + \frac{(d-2)^5 e^{2\sqrt{d}}}{2^{d-1}} + 1 \right) q^{-1} + 14q^{-2}.$$

On the other hand, if $p > 2$ and $s \leq \min \left\{ \binom{d+r-3}{r-1}, q^{r-1} \right\}$, then

$$|p_{r,d}[C_{\underline{a}} = s] - (1 - \mu_d)^{s-1} \mu_d| \leq d^2 2^d q^{-\frac{1}{2}} + (266 d^{d+5} e^{2\sqrt{d}-d} + 1) q^{-1}.$$

Proof. Suppose that $s \leq \min \left\{ \binom{d/2+r-1}{r-1}, q^{r-1} \right\}$. Then $\kappa_s \leq d/2$, and thus $1 \leq \kappa_i - 1 \leq d/2 - 1$ for $1 \leq i \leq s$. With notations as in Subsection 3.2, fix $1 \leq i \leq s$ and $h_j^* := (h_{d-\kappa_j,j}, \dots, h_{0,j}) \in \mathbb{F}_q^{d+1-\kappa_j}$ for $1 \leq j \leq i-1$. Denote $\mathbf{h}_{i-1}^* := (h_1^*, \dots, h_{i-1}^*)$, set $\psi_i^{\text{fix}}(\mathbf{h}_{i-1}^*) := (h_{d,i}, \dots, h_{d+1-\kappa_i,i})$ and consider the average cardinality $\mathcal{V}_d(\kappa_i, \psi_i^{\text{fix}}(\mathbf{h}_{i-1}^*))$ as in (3.13) or (3.14). By (3.15) we conclude that, for any \mathbf{h}_{i-1}^* with $\deg f_{h_i^*} = d$,

$$\left| \frac{\mathcal{V}_d(\kappa_i, \psi_i^{\text{fix}}(\mathbf{h}_{i-1}^*))}{q} - \mu_d \right| \leq \left(\frac{e^{-1}}{2} + \frac{(d-2)^5 e^{2\sqrt{d}}}{2^{d-2}} \right) q^{-1} + 7q^{-2}.$$

Further, defining \mathcal{T}_i as in the statement of Theorem 3.9 for $1 \leq i \leq s$, we obtain

$$|\mathcal{T}_i| \leq (1 - \mu_d)^{s-i-1} \mu_d \left(\left(\frac{e^{-1}}{2} + \frac{(d-2)^5 e^{2\sqrt{d}}}{2^{d-2}} \right) q^{-1} + 7q^{-2} \right) \quad (1 \leq i \leq s-1),$$

$$|\mathcal{T}_s| \leq \left(\frac{e^{-1}}{2} + \frac{(d-2)^5 e^{2\sqrt{d}}}{2^{d-2}} \right) q^{-1} + 7q^{-2}.$$

Therefore, the first assertion of the theorem follows from Theorem 3.9.

On the other hand, for $s \leq \min \left\{ \binom{d+r-3}{r-1}, q^{r-1} \right\}$ we have $\kappa_s \leq d-2$, and hence $\kappa_i - 1 \leq d-3$ for $1 \leq i \leq s$. Therefore, if $p > 2$, then (3.16) shows that

$$\left| \frac{\mathcal{V}_d(\kappa_i, \psi_i^{\text{fix}}(\mathbf{h}_{i-1}^*))}{q} - \mu_d \right| \leq d^2 2^{d-1} q^{-\frac{1}{2}} + 133 d^{d+5} e^{2\sqrt{d}-d} q^{-1}.$$

It follows that

$$|\mathcal{T}_i| \leq (1 - \mu_d)^{s-i-1} \mu_d (d^2 2^{d-1} q^{-\frac{1}{2}} + 133 d^{d+5} e^{2\sqrt{d}-d} q^{-1}) \quad (1 \leq i \leq s-1),$$

$$|\mathcal{T}_s| \leq d^2 2^{d-1} q^{-\frac{1}{2}} + 133 d^{d+5} e^{2\sqrt{d}-d} q^{-1}.$$

This readily implies the second assertion of the theorem. \square

We remark that the approach of the proof of Theorem 3.10 cannot be applied to estimate the probability that $s > s^* := \binom{d+r-3}{r-1}$ vertical strips are searched, since the behavior of the mapping $\Phi := \Phi_{\underline{a}} : \mathcal{F}_{r,d} \rightarrow \mathcal{F}_{1,d}^s$ of (3.1) may change significantly in this case. In what concerns “large” values of s , from Theorem 3.10 one easily deduces the following result.

Corollary 3.11. *With notations as in Theorem 3.10, for $s^* := \min \left\{ \binom{d+r-1}{r-1}, q^{r-1} \right\}$ we have*

$$p_{r,d}[C_{\underline{a}} > s^*] = (1 - \mu_d)^{s^*} + \mathcal{O}(q^{-1}).$$

On the other hand, if $p > 2$ and $s^ := \min \left\{ \binom{d+r-3}{r-1}, q^{r-1} \right\}$, then*

$$p_{r,d}[C_{\underline{a}} > s^*] = (1 - \mu_d)^{s^*} + \mathcal{O}(q^{-1/2}).$$

As $|1 - \mu_d| \leq 1/2$, from the expression of s^* in both cases it follows that the main term of this probability decreases exponentially with r and d .

4. PROBABILISTIC ANALYSIS OF THE SVS ALGORITHM

In this section we determine the average-case complexity of the SVS algorithm. This analysis relies on the probability distribution of the number of searches performed, which is the subject of the next section.

4.1. Probability distribution of the number of searches. Similarly to Section 2, for $s \geq 3$ we denote

$$\mathbf{F}_s := \{(\mathbf{a}_1, \dots, \mathbf{a}_s) \in \mathbb{F}_q^{r-1} \times \dots \times \mathbb{F}_q^{r-1} : \mathbf{a}_i \neq \mathbf{a}_j \text{ for } i \neq j\}, \quad N_s := |\mathbf{F}_s|,$$

and consider the random variable $C_s := C_{s,r,d} : \mathbf{F}_s \times \mathcal{F}_{r,d} \rightarrow \{1, \dots, s, \infty\}$ defined for $\underline{\mathbf{a}} := (\mathbf{a}_1, \dots, \mathbf{a}_s) \in \mathbf{F}_s$ and $F \in \mathcal{F}_{r,d}$ in the following way:

$$C_s(\underline{\mathbf{a}}, F) := \begin{cases} \min\{j : N_{1,d}(F(\mathbf{a}_j, X_r)) > 0\} & \text{if } \exists j \text{ with } N_{1,d}(F(\mathbf{a}_j, X_r)) > 0, \\ \infty & \text{otherwise.} \end{cases}$$

We consider the set $\mathbf{F}_s \times \mathcal{F}_{r,d}$ as before endowed with the uniform probability $P_s := P_{s,r,d}$ and analyze the probability $P_s[C_s = s]$. To link the probability spaces determined by $\mathbf{F}_s \times \mathcal{F}_{r,d}$ and P_s for $1 \leq s \leq q^{r-1}$, we have the following result.

Lemma 4.1. *Let $s > 1$ and let $\pi_s : \mathbf{F}_s \times \mathcal{F}_{r,d} \rightarrow \mathbf{F}_{s-1} \times \mathcal{F}_{r,d}$ be the mapping induced by the projection $\mathbf{F}_s \rightarrow \mathbf{F}_{s-1}$ on the first $s-1$ coordinates. If $\mathcal{S} \subset \mathbf{F}_{s-1} \times \mathcal{F}_{r,d}$, then $P_s[\pi_s^{-1}(\mathcal{S})] = P_{s-1}[\mathcal{S}]$.*

Proof. Note that

$$\begin{aligned} \pi_s^{-1}(\mathcal{S}) &= \bigcup_{F \in \mathcal{F}_{r,d}} \{(\mathbf{a}_1, \dots, \mathbf{a}_s) \in \mathbf{F}_s : (\mathbf{a}_1, \dots, \mathbf{a}_{s-1}, F) \in \mathcal{S}\} \times \{F\} \\ &= \bigcup_{F \in \mathcal{F}_{r,d}} \bigcup_{\substack{(\mathbf{a}_1, \dots, \mathbf{a}_{s-1}) \in \mathbf{F}_{s-1} : \\ (\mathbf{a}_1, \dots, \mathbf{a}_{s-1}, F) \in \mathcal{S}}} \{(\mathbf{a}_1, \dots, \mathbf{a}_{s-1})\} \times (\mathbb{F}_q^{r-1} \setminus \{\mathbf{a}_1, \dots, \mathbf{a}_{s-1}\}) \times \{F\}. \end{aligned}$$

It follows that

$$\begin{aligned} P_s[\pi_s^{-1}(\mathcal{S})] &= \frac{1}{N_s |\mathcal{F}_{r,d}|} \sum_{F \in \mathcal{F}_{r,d}} \sum_{\underline{\mathbf{a}} \in \mathbf{F}_{s-1} : (\underline{\mathbf{a}}, F) \in \mathcal{S}} (q^{r-1} - s + 1) \\ &= \frac{1}{N_{s-1} |\mathcal{F}_{r,d}|} \sum_{F \in \mathcal{F}_{r,d}} |\{\underline{\mathbf{a}} \in \mathbf{F}_{s-1} : (\underline{\mathbf{a}}, F) \in \mathcal{S}\}| = P_{s-1}[\mathcal{S}]. \end{aligned}$$

This proves the lemma. \square

According to the Kolmogorov extension theorem (see, e.g., [12, Chapter IV, Section 5, Extension Theorem]), the conditions of “consistency” of Lemma 4.1 imply that the probabilities P_s ($1 \leq s \leq q^{r-1}$) can be put in a unified framework. More precisely, we define $\mathbf{F} := \mathbf{F}_{q^{r-1}}$ and $P := P_{q^{r-1}}$. Then the probability measure P defined on \mathbf{F} allows us to interpret consistently all the results of this paper. In the same vein, the variables C_s ($1 \leq s \leq q^{r-1}$) can be naturally extended to a random variable $C : \mathbf{F} \times \mathcal{F}_{r,d} \rightarrow \mathbb{N} \cup \{\infty\}$. Consequently, we shall drop the subscript s from the notations P_s and C_s in what follows.

For the analysis of the probability distribution of the number of searches we express the probability $P[C = s]$ in terms of probabilities concerning the random variables $C_{\underline{\mathbf{a}}} := C_{\underline{\mathbf{a}},r,d} : \mathcal{F}_{r,d} \rightarrow \mathbb{N}$, $\underline{\mathbf{a}} \in \mathbf{F}_s$, which count the number of vertical strips that are searched when the choice for the first s vertical strips is $\underline{\mathbf{a}}$. As the result can be proved following the proof of Lemma 2.3 *mutatis mutandis*, we state it without proof.

Lemma 4.2. *We have*

$$P[C = s] = \frac{1}{N_s} \sum_{\underline{\mathbf{a}} \in \mathbf{F}_s} p_{r,d}[C_{\underline{\mathbf{a}}} = s].$$

In Theorem 3.10 we determine the asymptotic behavior of $p_{r,d}[C_{\underline{\mathbf{a}}} = s]$ for $\underline{\mathbf{a}} \in \mathbf{F}_s \setminus \mathbf{B}_s$, where $\mathbf{B}_s \subset \mathbf{F}_s$ is the set of (3.5). By (3.4) it follows that $|\mathbf{B}_s| = \mathcal{O}(q^{s(r-1)-1})$, where the \mathcal{O} -constant depends on s , d and r , but is independent of q . Now, to

estimate the probability $P[C = s]$, Lemma 4.2 implies

$$\begin{aligned} P[C = s] &= \frac{1}{N_s} \sum_{\underline{a} \in \mathcal{F}_s \setminus \mathcal{B}_s} p_{r,d}[C_{\underline{a}} = s] + \frac{1}{N_s} \sum_{\underline{a} \in \mathcal{B}_s} p_{r,d}[C_{\underline{a}} = s] \\ &= \frac{1}{N_s} \sum_{\underline{a} \in \mathcal{F}_s \setminus \mathcal{B}_s} p_{r,d}[C_{\underline{a}} = s] + \mathcal{O}(q^{-1}). \end{aligned}$$

As a consequence, from Theorem 3.10 we deduce the following result.

Theorem 4.3. *For $s \leq \binom{d/2+r-1}{r-1}$, we have*

$$P[C = s] = (1 - \mu_d)^{s-1} \mu_d + \mathcal{O}(q^{-1}).$$

On the other hand, if $p > 2$ and $s \leq \binom{d+r-3}{r-1}$, then

$$P[C = s] = (1 - \mu_d)^{s-1} \mu_d + \mathcal{O}(q^{-1/2}).$$

4.2. Average-case complexity. Now we are ready to determine the average-case complexity of the SVS algorithm.

Recall that, given $F \in \mathcal{F}_{r,d}$, the SVS algorithm successively generates a sequence $\underline{a} := (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{q^{r-1}}) \in \mathbb{F}_{q^{r-1}}$, and searches for \mathbb{F}_q -rational zeros of F in the vertical strips $\{\mathbf{a}_i\} \times \mathbb{F}_q$ for $1 \leq i \leq q^{r-1}$, until a zero of F is found or all the vertical strips are exhausted. As discussed in Section 1, the whole procedure requires at most $C_{\underline{a}}(F) \cdot \tau(d, r, q)$ arithmetic operations in \mathbb{F}_q , where $\tau(d, r, q) := \mathcal{O}^\sim(D + d \log_2 q)$ is the maximum number of arithmetic operations in \mathbb{F}_q necessary to perform a search in an arbitrary vertical strip.

The SVS algorithm has a probabilistic routine which searches for \mathbb{F}_q -rational zeros of elements of $\mathcal{F}_{1,d}$, which relies on r_d random choices of elements of \mathbb{F}_q , for certain $r_d \in \mathbb{N}$. We denote by $\Omega_d := \mathbb{F}_q^{r_d}$ the set of all such random choices and consider Ω_d endowed with the uniform probability, $\mathcal{F} \times \mathcal{F}_{r,d}$ with the (uniform) probability P of Section 4, and $\mathcal{F} \times \mathcal{F}_{r,d} \times \Omega_d$ with the product probability. Therefore, the cost of the SVS algorithm is represented by the random variable $X := X_{r,d} : \mathcal{F} \times \mathcal{F}_{r,d} \times \Omega_d \rightarrow \mathbb{N}_{\geq 0}$ which counts the number $X(\underline{a}, F, \omega)$ of arithmetic operations performed on input $F \in \mathcal{F}_{r,d}$, with the choice of vertical strips defined by \underline{a} and the choice ω for the parameters of the routine for univariate root finding.

We aim to determine the asymptotic behavior of the expected value of X , namely

$$E[X] := \frac{1}{|\mathcal{F}| |\mathcal{F}_{r,d}| |\Omega_d|} \sum_{(\underline{a}, F, \omega)} X(\underline{a}, F, \omega) \leq \frac{\tau(d, r, q)}{|\mathcal{F}| |\mathcal{F}_{r,d}|} \sum_{F \in \mathcal{F}_{r,d}} \sum_{\underline{a} \in \mathcal{F}} C(\underline{a}, F).$$

We first study the case $r > 2$, for which we have the following result.

Theorem 4.4. *Let $r > 2$ and $s^* := \binom{d/2+r-1}{r-1}$. Then the average-case complexity of the SVS algorithm is bounded in the following way:*

$$(4.1) \quad E[X] \leq \tau(d, r, q) (\mu_d^{-1} + d(1 - d^{-1})^{s^*}) + \mathcal{O}(q^{-1/2}),$$

where $\tau(d, r, q)$ is the cost of the search in a vertical strip.

Proof. Recall that an element of $\mathcal{F}_{r,d}$ is called relatively \mathbb{F}_q -irreducible if none of its irreducible factors over \mathbb{F}_q is absolutely irreducible. Consider the sets

$$A := \{F \in \mathcal{F}_{r,d} : F \text{ is relatively } \mathbb{F}_q\text{-irreducible}\}, \quad B := \mathcal{F}_{r,d} \setminus A.$$

We have

$$(4.2) \quad \sum_{F \in \mathcal{F}_{r,d}} \sum_{\underline{a} \in \mathcal{F}} C(\underline{a}, F) = \sum_{F \in A} \sum_{\underline{a} \in \mathcal{F}} C(\underline{a}, F) + \sum_{F \in B} \sum_{\underline{a} \in \mathcal{F}} C(\underline{a}, F).$$

By [16, Corollary 6.7], it follows that $|A|/|\mathcal{F}_{r,d}| = \mathcal{O}(q^{\frac{-r(r-1)}{2}})$. Hence, we obtain

$$(4.3) \quad \frac{1}{|\mathcal{F}| |\mathcal{F}_{r,d}|} \sum_{F \in A} \sum_{\underline{a} \in \mathcal{F}} C(\underline{a}, F) \leq \frac{q^{r-1}}{|\mathcal{F}_{r,d}|} |A| = \mathcal{O}(q^{\frac{(r-1)(2-r)}{2}}) = \mathcal{O}(q^{-1}).$$

Next we study the second term in the right-hand side of (4.2). We have

$$\frac{1}{|\mathbf{F}||\mathcal{F}_{r,d}|} \sum_{F \in B} \sum_{\underline{a} \in \mathbf{F}} C(\underline{a}, F) = \frac{1}{|\mathcal{F}_{r,d}|} \sum_{F \in B} \sum_{s=1}^{q^{r-1}} s \frac{|\{\underline{a} \in \mathbf{F} : C(\underline{a}, F) = s\}|}{|\mathbf{F}|}.$$

From the conditions of consistency of Lemma 4.1, it follows that

$$\begin{aligned} \frac{1}{|\mathbf{F}||\mathcal{F}_{r,d}|} \sum_{F \in B} \sum_{\underline{a} \in \mathbf{F}} C(\underline{a}, F) &= \frac{|B|}{|\mathcal{F}_{r,d}|} \sum_{s=1}^{q^{r-1}} s \frac{1}{|B|} \sum_{F \in B} \frac{|\{\underline{a} \in \mathbf{F}_s : C(\underline{a}, F) = s\}|}{|\mathbf{F}_s|} \\ &= \frac{|B|}{|\mathcal{F}_{r,d}|} \sum_{s=1}^{q^{r-1}} s P_{\mathbf{F} \times B}[C = s], \end{aligned}$$

where $P_{\mathbf{F} \times B}$ denotes the uniform probability in $\mathbf{F} \times B$.

For $s \leq s^*$, Theorem 4.3 allows us to estimate the probability of $[C = s]$. Therefore, we decompose the sum above in the following way:

$$\begin{aligned} \sum_{s=1}^{q^{r-1}} s P_{\mathbf{F} \times B}[C = s] &= \sum_{s=1}^{s^*} s P_{\mathbf{F} \times B}[C = s] + (s^* + 1) \sum_{s=s^*+1}^{q^{r-1}} P_{\mathbf{F} \times B}[C = s] \\ &\quad + \sum_{s=s^*+2}^{q^{r-1}} (s - s^* - 1) P_{\mathbf{F} \times B}[C = s] \\ (4.4) \quad &= \sum_{s=1}^{s^*} s P_{\mathbf{F} \times B}[C = s] + (s^* + 1) P_{\mathbf{F} \times B}[C \geq s^* + 1] + \sum_{s=s^*+2}^{q^{r-1}} P_{\mathbf{F} \times B}[C \geq s]. \end{aligned}$$

First we estimate the sum S_1 of the first two terms in the right-hand of (4.4). Arguing as in Lemma 4.2, we see that

$$P_{\mathbf{F} \times B}[C = s] = \frac{1}{|\mathbf{F}_s|} \sum_{\underline{a} \in \mathbf{F}_s} p_B[C_{\underline{a}} = s].$$

From Theorem 4.3 and Corollary 3.11 we have

$$\begin{aligned} S_1 &= \sum_{s=1}^{s^*} s(\mu_d(1 - \mu_d)^{s-1} + \mathcal{O}(q^{-1})) + (s^* + 1)(1 - \mu_d)^{s^*} + \mathcal{O}(q^{-1}) \\ &= \mu_d \sum_{s=1}^{s^*} s(1 - \mu_d)^{s-1} + (s^* + 1)(1 - \mu_d)^{s^*} + \mathcal{O}(q^{-1}). \end{aligned}$$

Taking into account that $\sum_{n \geq 1} n z^{n-1} = 1/(1 - z)^2$ for any $|z| \leq 1$, we obtain

$$(4.5) \quad S_1 = \frac{1}{\mu_d} - \mu_d \sum_{s \geq s^*+1} s(1 - \mu_d)^{s-1} + (s^* + 1)(1 - \mu_d)^{s^*} + \mathcal{O}(q^{-1}) = \frac{1}{\mu_d} + \mathcal{O}(q^{-1}),$$

where the last inequality follows from the identity $\sum_{s \geq s^*+1} s z^{s-1} = z^{s^*} (s^* + 1 - z s^*)/(1 - z)^2$, which holds for any $|z| < 1$ (see, e.g., [17, §2.3]).

Next, we estimate the second sum S_2 of the right-hand of (4.4). Observe that

$$p_B[C_{\underline{a}} \geq s] = p_B[F \in B : N_{1,d}(F(\mathbf{a}_i, X_r)) = 0 \ (1 \leq i \leq s - 1)].$$

Hence,

$$\begin{aligned}
S_2 &\leq \frac{1}{|B|} \sum_{s=s^*+2}^{q^{r-1}} \frac{1}{|F_s|} \sum_{(\underline{a}, \mathbf{a}_s) \in F_{s-1} \times \mathbb{F}_q^{r-1}} |\{F \in B : N_{1,d}(F(\mathbf{a}_i, X_r)) = 0 \ (1 \leq i \leq s-1)\}| \\
&\leq \frac{q^{r-1}}{|B|} \sum_{s=s^*+2}^{q^{r-1}} \frac{1}{q^{r-1} - (s-1)} \sum_{\underline{a} \in F_{s-1}} \sum_{\substack{F \in B \\ N_{1,d}(F(\mathbf{a}_i, X_r)) = 0 \ (1 \leq i \leq s-1)}} \frac{1}{|F_{s-1}|} \\
&\leq \frac{q^{r-1}}{|B|} \sum_{s=s^*+2}^{q^{r-1}} \frac{1}{q^{r-1} - (s-1)} \sum_{F \in B} P_{F_{s-1}}[N_{1,d} = 0],
\end{aligned}$$

where $P_{F_{s-1}}[N_{1,d} = 0] := P_{F_{s-1}}[\{\underline{a} \in F_{s-1} : N_{1,d}(F(\mathbf{a}_i, X_r)) = 0, \ 1 \leq i \leq s-1\}]$. As $N_{1,d} = 0$ follows an hypergeometric distribution, the probability $P_{F_{s-1}}[N_{1,d} = 0]$ can be expressed in the following way (see, e.g., [11, Chapter 6]):

$$P_{F_{s-1}}[N_{1,d} = 0] = \frac{\binom{q^{r-1} - NS(F)}{s-1}}{\binom{q^{r-1}}{s-1}}.$$

We deduce that

$$(4.6) \quad S_2 \leq \frac{1}{|B|} \sum_{s=s^*+2}^{q^{r-1}} \sum_{F \in B} \left(1 - \frac{NS(F) - 1}{q^{r-1} - 1}\right)^{s-1}.$$

Fix $F \in B$. Then F has at least an absolutely irreducible factor defined over \mathbb{F}_q . Hence, for $q > d^4$, by [6, Theorem 5.2] it follows that $NS(F) \geq \frac{q^{r-1}}{d}(1 - \alpha)$, with $\alpha := d^2 q^{-1/2}$. This implies

$$1 - \frac{NS(F) - 1}{q^{r-1} - 1} = 1 - \frac{1 - \alpha}{d} + \mathcal{O}(q^{1-r}).$$

Combining this inequality with (4.6) we conclude that

$$\begin{aligned}
S_2 &\leq \frac{1}{|B|} \sum_{s=s^*+2}^{q^{r-1}} \sum_{F \in B} (1 - (1 - \alpha)d^{-1} + \mathcal{O}(q^{1-r}))^{s-1} \\
&= \sum_{s=s^*+2}^{q^{r-1}} (1 - (1 - \alpha)d^{-1} + \mathcal{O}(q^{1-r}))^{s-1} \\
&= \frac{(1 - (1 - \alpha)d^{-1})^{s^*+1}}{(1 - \alpha)d^{-1}} + \mathcal{O}(q^{1-r}) = d(1 - d^{-1})^{s^*+1} + \mathcal{O}(q^{-1/2}).
\end{aligned}$$

Combining (4.2), (4.3) and (4.5) with this inequality, we deduce (4.1). \square

Since $s^* > d^2/4$, the term $d(1 - d^{-1})^{s^*+1}$ tends to zero as d and r grow, and therefore the right-hand side of (4.1) behaves as $\mu_d^{-1} \tau(d, r, q)$. We may paraphrase this as saying that, on average, at most $\mu_d^{-1} \approx 1.58 \dots$ vertical strips are searched until an \mathbb{F}_q -rational point of the input polynomial is obtained. For perspective, we remark that the probabilistic algorithms of [15] (for bivariate polynomials) and [5] and [20] (for r -variate polynomials) propose d searches in order to achieve a probability of success greater than $1/2$.

Now we analyze the average-case complexity $E[X]$ for $r = 2$, that is,

$$E[X] := \frac{1}{|F||\mathcal{F}_{2,d}||\Omega_d|} \sum_{(\underline{a}, F, \omega)} X(\underline{a}, F, \omega) \leq \frac{\tau(d, r, q)}{|F||\mathcal{F}_{2,d}|} \sum_{F \in \mathcal{F}_{r,d}} \sum_{\underline{a} \in F} C(\underline{a}, F).$$

For a real $0 < \alpha < 1$ to be determined, we consider the subsets

$$\begin{aligned}
A &:= \{F \in \mathcal{F}_{2,d} : NS(F) \leq (1 - \alpha)NS(2, d)\}, \\
B &:= \{F \in \mathcal{F}_{2,d} : NS(F) > (1 - \alpha)NS(2, d)\},
\end{aligned}$$

where $NS(F)$ is the number of vertical strips on which F has \mathbb{F}_q -rational zeros, and $NS(2, d)$ is the average number of such vertical strips. We have

$$(4.7) \quad \sum_{F \in \mathcal{F}_{2,d}} \sum_{\underline{a} \in \mathcal{F}} C(\underline{a}, F) = \sum_{F \in A} \sum_{\underline{a} \in \mathcal{F}} C(\underline{a}, F) + \sum_{F \in B} \sum_{\underline{a} \in \mathcal{F}} C(\underline{a}, F).$$

To estimate the first term of the right-hand of (4.7), we start with an estimate for $|A|$. For this purpose, according to Lemma 5.1 and Proposition 5.2 below, the mean $NS(2, d)$ and the variance $NS_2(2, d)$ of $NS(\cdot)$ have the asymptotic behavior $NS(2, d) = \mu_d q + \mathcal{O}(1)$ and $NS_2(2, d) = ((d!)^{-2} + \mu_d(1 - \mu_d))q + \mathcal{O}(1)$ respectively. Then the Chebyshev inequality (see Corollary 5.3 below) implies

$$|A| \leq \left(\frac{1}{(\alpha \mu_d d!)^2} + \frac{1 - \mu_d}{\alpha^2 \mu_d} \right) q^{\dim \mathcal{F}_{2,d}-1} + \mathcal{O}(q^{\dim \mathcal{F}_{2,d}-2}).$$

It follows that

$$(4.8) \quad \frac{1}{|\mathcal{F}| |\mathcal{F}_{2,d}|} \sum_{F \in A} \sum_{\underline{a} \in \mathcal{F}} C(\underline{a}, F) \leq \frac{|A|q}{|\mathcal{F}_{2,d}|} \leq \left(\frac{1}{(\alpha \mu_d d!)^2} + \frac{1 - \mu_d}{\alpha^2 \mu_d} \right) + \mathcal{O}(q^{-1}).$$

Next we study the second sum in the right-hand side of (4.7). Arguing as in the case $r > 2$, for $s^* := d/2 + 1$ we obtain

$$\frac{1}{|\mathcal{F}| |\mathcal{F}_{2,d}|} \sum_{F \in B} \sum_{\underline{a} \in \mathcal{F}} C(\underline{a}, F) \leq \frac{1}{\mu_d} + \frac{1}{|B|} \sum_{s=s^*+2}^q \sum_{F \in B} \left(1 - \frac{NS(F) - 1}{q - 1} \right)^{s-1} + \mathcal{O}(q^{-1}).$$

Fix $F \in B$. By definition $NS(F) > (1 - \alpha)NS(2, d)$ and, according to Lemma 5.1 below, we have $NS(2, d) = \mu_d q + \mathcal{O}(1)$. Hence, we obtain

$$1 - \frac{NS(F) - 1}{q - 1} \leq 1 - (1 - \alpha)\mu_d + \mathcal{O}(q^{-1}).$$

Therefore,

$$\frac{1}{|B|} \sum_{s=s^*+2}^q \sum_{F \in B} \left(1 - \frac{NS(F) - 1}{q - 1} \right)^{s-1} \leq \frac{(1 - (1 - \alpha)\mu_d)^{s^*+1}}{(1 - \alpha)\mu_d} + \mathcal{O}(q^{-1}).$$

Combining (4.7) and (4.8) with this inequality, we conclude that

$$E[X] \leq \tau(d, r, q) \left(\frac{1}{\alpha^2} \left(\frac{1 - \mu_d}{\mu_d} + \frac{1}{(d!)^2 \mu_d^2} \right) + \frac{1}{\mu_d} + (1 - (1 - \alpha)\mu_d)^{s^*+1} \right) + \mathcal{O}(q^{-1}).$$

Fixing $\alpha^* := 1 - 1/\sqrt{s^*}$, we obtain the following result.

Theorem 4.5. *Let $r := 2$, $s^* := d/2 + 1$ and $\alpha^* := 1 - 1/\sqrt{s^*}$. The average-case complexity of the SVS algorithm is bounded in the following way:*

$$E[X] \leq \tau(d, r, q) \left(\frac{1}{\alpha^{*2}} \left(\frac{1 - \mu_d}{\mu_d} + \frac{1}{(d!)^2 \mu_d^2} \right) + \frac{1}{\mu_d} + \left(1 - \frac{\mu_d}{\sqrt{s^*}} \right)^{s^*+1} \right) + \mathcal{O}(q^{-1}),$$

where $\tau(d, r, q)$ is the cost of the search in a vertical strip.

As d grows, the quantity s^* tends to infinity and the expression parenthesized in $E[X]$ tends to $(2 - \mu_d)/\mu_d \approx 2.16 \dots$. This is an upper bound for the number of vertical strips that are searched on average for $r = 2$.

5. ON THE PROBABILITY DISTRIBUTION OF THE OUTPUTS

This section is devoted to the analysis of the probability distribution of the outputs of the SVS algorithm. For this purpose, following [3] (see also [2]), we use the concept of Shannon entropy. For $F \in \mathcal{F}_{r,d}$, denote $Z(F) := \{\mathbf{x} \in \mathbb{F}_q^r : F(\mathbf{x}) = \mathbf{0}\}$ and $N(F) := |Z(F)|$. We define a Shannon entropy H_F associated with F as

$$(5.1) \quad H_F := \sum_{\mathbf{x} \in Z(F)} -P_{\mathbf{x},F} \log(P_{\mathbf{x},F}),$$

where $P_{\mathbf{x},F}$ is the probability that the SVS algorithm outputs \mathbf{x} on input F and \log denotes the natural logarithm. It is well-known that $H_F \leq \log N(F)$, and equality

holds if and only if $P_{\mathbf{x},F} = 1/N(F)$ for every $\mathbf{x} \in Z(F)$. We shall consider the average entropy when F runs through all the elements of $\mathcal{F}_{r,d}$, namely

$$(5.2) \quad H := \frac{1}{|\mathcal{F}_{r,d}|} \sum_{F \in \mathcal{F}_{r,d}} H_F.$$

For an “ideal” algorithm for the search of \mathbb{F}_q -rational zeros of elements of $\mathcal{F}_{r,d}$, from the point of view of the probability distribution of outputs, and $F \in \mathcal{F}_{r,d}$, the probability $P_{\mathbf{x},F}^{\text{ideal}}$ that a given $\mathbf{x} \in Z(F)$ occurs as output is equal to $1/N(F)$. As a consequence, according to the definition (5.1), the corresponding entropy is

$$H_F^{\text{ideal}} := \sum_{\mathbf{x} \in Z(F)} -P_{\mathbf{x},F}^{\text{ideal}} \log(P_{\mathbf{x},F}^{\text{ideal}}) = \sum_{\mathbf{x} \in Z(F)} \frac{\log N(F)}{N(F)} = \log N(F).$$

By the concavity of the function $x \mapsto \log x$ and (1.1), we conclude that

$$(5.3) \quad H^{\text{ideal}} := \frac{1}{|\mathcal{F}_{r,d}|} \sum_{F \in \mathcal{F}_{r,d}} H_F^{\text{ideal}} \leq \log \left(\frac{\sum_{F \in \mathcal{F}_{r,d}} N(F)}{|\mathcal{F}_{r,d}|} \right) = \log(q^{r-1}),$$

where the last identity is due to (1.1). In our analysis below, we shall exhibit a lower bound on the average entropy H which nearly matches this upper bound.

5.1. On the number of vertical strips. A critical point in the study of the behavior of H is the analysis of the probability distribution of the random variable $NS : \mathcal{F}_{r,d} \rightarrow \mathbb{Z}_{\geq 0}$ which counts the number of vertical strips with \mathbb{F}_q -rational zeros of the elements of $\mathcal{F}_{r,d}$.

Recall that $VS(F)$ denotes the set of vertical strips where each $F \in \mathcal{F}_{r,d}$ has \mathbb{F}_q -rational zeros and $NS(F)$ is its cardinality, that is,

$$VS(F) := \{\mathbf{a} \in \mathbb{F}_q^{r-1} : (\exists x_r \in \mathbb{F}_q) F(\mathbf{a}, x_r) = 0\}, \quad NS(F) := |VS(F)|.$$

We start considering the average number of vertical strips in $\mathcal{F}_{r,d}$, namely

$$NS(r, d) := \frac{1}{|\mathcal{F}_{r,d}|} \sum_{F \in \mathcal{F}_{r,d}} NS(F).$$

According to (2.1), we have $NS(r, d) = q^{r-1}P[C = 1]$. Therefore, as an immediate consequence of Theorem 2.1 and Corollary 2.2 we have the following result.

Lemma 5.1. *The number $NS(r, d)$ satisfies*

$$\begin{aligned} NS(r, d) &= \sum_{k=1}^d (-1)^{k-1} \binom{q}{k} q^{r-1-k} + (-1)^d \binom{q-1}{d} q^{r-d-2} \\ &= \mu_d q^{r-1} + \mathcal{O}(q^{r-2}). \end{aligned}$$

Next we determine the variance $NS_2(r, d)$ of the random variable $NS(\cdot)$, that is,

$$NS_2(r, d) := \frac{1}{|\mathcal{F}_{r,d}|} \sum_{F \in \mathcal{F}_{r,d}} (NS(F) - NS(r, d))^2 = \frac{1}{|\mathcal{F}_{r,d}|} \sum_{F \in \mathcal{F}_{r,d}} NS(F)^2 - NS(r, d)^2.$$

Proposition 5.2. *The variance $NS_2(r, d)$ satisfies*

$$NS_2(r, d) = \frac{1}{(d!)^2} q^{2r-3} + \mu_d(1 - \mu_d) q^{r-1} + \mathcal{O}(q^{2r-4}).$$

Proof. Recall the notations $F_2 := (\mathbb{F}_q^{r-1})^2 \setminus \{(\mathbf{a}, \mathbf{a}) : \mathbf{a} \in \mathbb{F}_q^{r-1}\}$ and $N_2 := |F_2|$. Fix $F \in \mathcal{F}_{r,d}$. We have

$$NS(F)^2 = \left| \bigcup_{x, y \in \mathbb{F}_q} \{(\mathbf{a}_1, \mathbf{a}_2) \in (\mathbb{F}_q^{r-1})^2 : F(\mathbf{a}_1, x) = F(\mathbf{a}_2, y) = 0\} \right|.$$

Then the inclusion–exclusion principle implies

$$\begin{aligned} \sum_{F \in \mathcal{F}_{r,d}} NS(F)^2 &= \sum_{F \in \mathcal{F}_{r,d}} \sum_{j=1}^q \sum_{k=1}^q (-1)^{j+k} \sum_{\mathcal{X}_j \subset \mathbb{F}_q} \sum_{\mathcal{Y}_k \subset \mathbb{F}_q} \mathcal{S}(\mathcal{X}_j, \mathcal{Y}_k) \\ &= \sum_{j=1}^q \sum_{k=1}^q (-1)^{j+k} \sum_{\mathcal{X}_j \subset \mathbb{F}_q} \sum_{\mathcal{Y}_k \subset \mathbb{F}_q} \sum_{F \in \mathcal{F}_{r,d}} \mathcal{S}(\mathcal{X}_j, \mathcal{Y}_k), \end{aligned}$$

where \mathcal{X}_j and \mathcal{Y}_k run through all the subsets of \mathbb{F}_q of cardinality j and k , respectively, and, for arbitrary subsets $\mathcal{X} \subset \mathbb{F}_q$ and $\mathcal{Y} \subset \mathbb{F}_q$,

$$\mathcal{S}(\mathcal{X}, \mathcal{Y}) := \left| \{(\mathbf{a}_1, \mathbf{a}_2) \in (\mathbb{F}_q^{r-1})^2 : (\forall x \in \mathcal{X})(\forall y \in \mathcal{Y}) F(\mathbf{a}_1, x) = 0, F(\mathbf{a}_2, y) = 0\} \right|.$$

For $\underline{\mathbf{a}} := (\mathbf{a}_1, \mathbf{a}_2) \in (\mathbb{F}_q^{r-1})^2$ and subsets $\mathcal{X} \subset \mathbb{F}_q$ and $\mathcal{Y} \subset \mathbb{F}_q$, denote

$$\mathcal{S}_{\underline{\mathbf{a}}}(\mathcal{X}, \mathcal{Y}) := \{F \in \mathcal{F}_{r,d} : (\forall x \in \mathcal{X})(\forall y \in \mathcal{Y}) F(\mathbf{a}_1, x) = 0, F(\mathbf{a}_2, y) = 0\}.$$

It follows that

$$\begin{aligned} \sum_{F \in \mathcal{F}_{r,d}} NS(F)^2 &= \sum_{j=1}^q \sum_{k=1}^q (-1)^{j+k} \sum_{\mathcal{X}_j \subset \mathbb{F}_q} \sum_{\mathcal{Y}_k \subset \mathbb{F}_q} \sum_{\underline{\mathbf{a}} \in (\mathbb{F}_q^{r-1})^2} |\mathcal{S}_{\underline{\mathbf{a}}}(\mathcal{X}_j, \mathcal{Y}_k)| \\ &= \sum_{\underline{\mathbf{a}} \in (\mathbb{F}_q^{r-1})^2} \sum_{j=1}^q \sum_{k=1}^q (-1)^{j+k} \sum_{\mathcal{X}_j \subset \mathbb{F}_q} \sum_{\mathcal{Y}_k \subset \mathbb{F}_q} |\mathcal{S}_{\underline{\mathbf{a}}}(\mathcal{X}_j, \mathcal{Y}_k)| =: \sum_{\underline{\mathbf{a}} \in (\mathbb{F}_q^{r-1})^2} N_{\underline{\mathbf{a}},2}, \end{aligned}$$

where $N_{\underline{\mathbf{a}},2}$ is defined as in (2.6). If $\underline{\mathbf{a}} \in \mathbf{F}_2$, then the claim in the proof of Proposition 2.4 asserts that

$$\frac{N_{\underline{\mathbf{a}},2}}{|\mathcal{F}_{r,d}|} = (P[C=1])^2 + \frac{q-1}{q^{2d+2}} \binom{q-1}{d}^2.$$

On the other hand, for $(\mathbf{a}, \mathbf{a}) \in (\mathbb{F}_q^{r-1})^2 \setminus \mathbf{F}_2$, by elementary calculations we see that

$$N_{(\mathbf{a}, \mathbf{a}),2} := \sum_{j=1}^q \sum_{k=1}^q (-1)^{j+k} \sum_{\mathcal{X}_j \subset \mathbb{F}_q} \sum_{\mathcal{Y}_k \subset \mathbb{F}_q} |\mathcal{S}_{(\mathbf{a}, \mathbf{a})}(\mathcal{X}_j, \mathcal{Y}_k)| = \sum_{j=1}^q (-1)^{j-1} \sum_{\mathcal{X}_j \subset \mathbb{F}_q} |\mathcal{S}_{\mathbf{a}}(\mathcal{X}_j)|,$$

where $\mathcal{S}_{\mathbf{a}}(\mathcal{Z}) := \{F \in \mathcal{F}_{r,d} : (\forall z \in \mathcal{Z}) F(\mathbf{a}, z) = 0\}$ for any subset $\mathcal{Z} \subset \mathbb{F}_q$. Thus,

$$\begin{aligned} \frac{1}{|\mathcal{F}_{r,d}|} \sum_{F \in \mathcal{F}_{r,d}} NS(F)^2 &= \sum_{\underline{\mathbf{a}} \in \mathbf{F}_2} \frac{N_{\underline{\mathbf{a}},2}}{|\mathcal{F}_{r,d}|} + \frac{1}{|\mathcal{F}_{r,d}|} \sum_{\mathbf{a} \in \mathbb{F}_q^{r-1}} \sum_{j=1}^q (-1)^{j-1} \sum_{\mathcal{X}_j \subset \mathbb{F}_q} |\mathcal{S}_{\mathbf{a}}(\mathcal{X}_j)| \\ &= N_2 \left((q^{1-r} NS(r, d))^2 + \frac{q-1}{q^{2d+2}} \binom{q-1}{d}^2 \right) + \sum_{F \in \mathcal{F}_{r,d}} \frac{NS(F)}{|\mathcal{F}_{r,d}|}. \end{aligned}$$

The statement of the proposition follows easily from Lemma 5.1. \square

By the Chebyshev inequality we obtain a lower bound on the number of $F \in \mathcal{F}_{r,d}$ for which $NS(F)$ differs a certain proportion from the expected value $NS(r, d)$.

Corollary 5.3. *For $0 < \alpha < 1$, the number $A(\alpha)$ of $F \in \mathcal{F}_{r,d}$ for which $NS(F) \leq (1 - \alpha)NS(r, d)$ is bounded as*

$$A(\alpha) \leq \frac{1}{(\alpha \mu_d d!)^2} q^{\dim \mathcal{F}_{r,d}-1} + \frac{1}{\alpha^2} \frac{1 - \mu_d}{\mu_d} q^{\dim \mathcal{F}_{r,d}-r+1} + \mathcal{O}(q^{\dim \mathcal{F}_{r,d}-2}).$$

Proof. By Lemma 5.1 and Proposition 5.2, the Chebyshev inequality implies

$$p_{r,d}(|NS(F) - NS(r, d)| \geq \alpha NS(r, d)) \leq \frac{NS_2(r, d)}{\alpha^2 NS(r, d)^2}.$$

Taking into account that

$$\frac{NS_2(r, d)}{\alpha^2 NS(r, d)^2} = \frac{1}{(\alpha \mu_d d!)^2} q^{-1} + \frac{1 - \mu_d}{\alpha^2 \mu_d} q^{1-r} + \mathcal{O}(q^{-2}),$$

the corollary readily follows. \square

5.2. A lower bound for the entropy. In order to analyze the Shannon entropy (5.2), it is necessary to determine the probability $P_{\mathbf{x},F}$ that an element $\mathbf{x} := (\mathbf{a}, x) \in \mathbb{F}_q^r$ occurs as output on input $F \in \mathcal{F}_{r,d}$.

Given an input polynomial $F \in \mathcal{F}_{r,d}$, and the vertical strip defined by an element $\mathbf{a} \in \mathbb{F}_q^{r-1}$, the SVS algorithm proceeds to search for \mathbb{F}_q -rational zeros of the univariate polynomial $f := \gcd(F(\mathbf{a}, T), T^q - T)$. If this search is done using the randomized algorithm of Cantor and Zassenhaus (see [7]), then all the \mathbb{F}_q^\times -rational zeros of f are equiprobable (see, e.g., [14, Section 14.3]). The algorithm can be easily modified so that all \mathbb{F}_q -rational zeros of f are equiprobable. In the sequel we shall assume that the search of roots in \mathbb{F}_q of elements of $\mathcal{F}_{1,d}$ is performed using a randomized algorithm for which all outputs are equiprobable.

For the analysis of the distribution of outputs, we denote as before by $\Omega_d := \mathbb{F}_q^{r,d}$ the set of all possible random choices of elements of \mathbb{F}_q made by the routine for univariate root finding. We consider Ω_d to be endowed with the uniform probability, $\mathbf{F} \times \mathcal{F}_{r,d}$ with the probability measure P of Section 4, and $\mathbf{F} \times \mathcal{F}_{r,d} \times \Omega_d$ with the product probability $P \times P_{\Omega_d}$. Finally, we shall consider probabilities related to the random variable $C_{\text{out}} : \mathbf{F} \times \mathcal{F}_{r,d} \times \Omega_d \rightarrow \mathbb{F}_q^r \cup \{\emptyset\}$ defined in the following way: for a triple $(\underline{\mathbf{a}}, F, \gamma) \in \mathbf{F} \times \mathcal{F}_{r,d} \times \Omega_d$, if F has an \mathbb{F}_q -rational zero on any of the vertical strips defined by $\underline{\mathbf{a}}$, and \mathbf{a}_j is the first vertical strip with this property, then $C_{\text{out}}(\underline{\mathbf{a}}, F, \gamma) := (\mathbf{a}_j, x)$, where $x \in \mathbb{F}_q$ is the zero of $F(\mathbf{a}_j, T)$ computed by the root-finding routine determined by the random choice γ . Otherwise, we define $C_{\text{out}}(\underline{\mathbf{a}}, F, \gamma) := \emptyset$. In these terms, the probability $P_{\mathbf{x},F}$ that an element $\mathbf{x} := (\mathbf{a}, x) \in \mathbb{F}_q^r$ occurs as output on input $F \in \mathcal{F}_{r,d}$ may be expressed as the conditional probability $P \times P_{\Omega_d}[C_{\text{out}} = \mathbf{x}|F]$, namely

$$P_{\mathbf{x},F} = P \times P_{\Omega_d}[C_{\text{out}} = \mathbf{x}|F] := \frac{P \times P_{\Omega_d}[\{C_{\text{out}} = \mathbf{x}\} \cap (\mathbf{F} \times \{F\} \times \Omega_d)]}{P \times P_{\Omega_d}[\mathbf{F} \times \{F\} \times \Omega_d]}.$$

Now we are ready to determine $P_{\mathbf{x},F}$. For this purpose, we denote by $N_{\mathbf{a}}(F)$ the number of \mathbb{F}_q -rational zeros of F in the vertical strip defined by \mathbf{a} , i.e.,

$$N_{\mathbf{a}}(F) := |\{x \in \mathbb{F}_q : F(\mathbf{a}, x) = 0\}|.$$

We have the following result.

Lemma 5.4. *Let $F \in \mathcal{F}_{r,d}$ and $\mathbf{x} := (\mathbf{a}, x) \in Z(F)$. Then*

$$P_{\mathbf{x},F} = \frac{1}{NS(F) N_{\mathbf{a}}(F)}.$$

Proof. If \mathbf{x} occurs as output at the j th step, then the SVS algorithm must have chosen elements $\mathbf{a}_1, \dots, \mathbf{a}_{j-1}$ for the first $j-1$ searches such that $N_{\mathbf{a}_k}(F) = 0$ for $1 \leq k \leq j-1$, and the element \mathbf{a} for the j th search. Finally, the routine for finding roots of $F(\mathbf{a}, T)$ must output x , which occurs with probability $1/N_{\mathbf{a}}(F)$.

Recall that the element $\mathbf{a}_j \in \mathbb{F}_q^{r-1}$ for the j th search is randomly chosen among the elements of $\mathbb{F}_q^{r-1} \setminus \{\mathbf{a}_1, \dots, \mathbf{a}_{j-1}\}$ with equiprobability. Therefore, if \mathbf{a} arises as the choice for the j th step, then the SVS algorithm must have chosen pairwise-distinct elements $\mathbf{a}_1, \dots, \mathbf{a}_{j-1} \in \mathbb{F}_q^{r-1} \setminus NS(F)$ for the first $j-1$ searches. The probability of these choices is

$$\begin{aligned} P(N_{\mathbf{a}_1}(F) = 0, \dots, N_{\mathbf{a}_{j-1}}(F) = 0, \mathbf{a}_j = \mathbf{a}|F) &= \prod_{k=0}^{j-2} \left(1 - \frac{NS(F)}{q^{r-1} - k}\right) \cdot \frac{1}{q^{r-1} - j + 1} \\ &= \frac{1}{q^{r-1}} \frac{\binom{q^{r-1} - NS(F)}{j-1}}{\binom{q^{r-1}-1}{j-1}}. \end{aligned}$$

As there are $q^{r-1} - NS(F)$ elements $\mathbf{b} \in \mathbb{F}_q^{r-1}$ with $N_{\mathbf{b}}(F) = 0$, the algorithm performs at most $q^{r-1} - NS(F) + 1$ searches. Finally, when \mathbf{a} is chosen, the probability to find x as the \mathbb{F}_q -rational zero of $F(\mathbf{a}, T)$ is equal to $1/N_{\mathbf{a}}(F)$. It

follows that

$$\begin{aligned} P_{\mathbf{x},F} &= \sum_{j=1}^{q^{r-1}-NS(F)+1} P(N_{\mathbf{a}_1}(F) = 0, \dots, N_{\mathbf{a}_{j-1}}(F) = 0, \mathbf{a}_j = \mathbf{a}|F) \cdot \frac{1}{N_{\mathbf{a}}(F)} \\ &= \frac{1}{q^{r-1}N_{\mathbf{a}}(F)} \sum_{j=0}^{q^{r-1}-NS(F)} \frac{\binom{q^{r-1}-NS(F)}{j}}{\binom{q^{r-1}-1}{j}}. \end{aligned}$$

According to, e.g., [17, §5.2, Problem 1],

$$\sum_{j=0}^{q^{r-1}-NS(F)} \frac{\binom{q^{r-1}-NS(F)}{j}}{\binom{q^{r-1}-1}{j}} = \frac{q^{r-1}}{NS(F)}.$$

We conclude that

$$P_{\mathbf{x},F} = \frac{1}{q^{r-1}N_{\mathbf{a}}(F)} \frac{q^{r-1}}{NS(F)} = \frac{1}{NS(F)N_{\mathbf{a}}(F)}.$$

This completes the proof of the lemma. \square

For any $F \in \mathcal{F}_{r,d}$, consider the entropy

$$(5.4) \quad H_F = \sum_{(\mathbf{a},x) \in Z(F)} \frac{\log(NS(F)N_{\mathbf{a}}(F))}{NS(F)N_{\mathbf{a}}(F)}.$$

We aim to determine the asymptotic behavior of the average entropy

$$H := \frac{1}{|\mathcal{F}_{r,d}|} \sum_{F \in \mathcal{F}_{r,d}} H_F = \frac{1}{|\mathcal{F}_{r,d}|} \sum_{F \in \mathcal{F}_{r,d}} \sum_{(\mathbf{a},x) \in Z(F)} \frac{\log(NS(F)N_{\mathbf{a}}(F))}{NS(F)N_{\mathbf{a}}(F)}.$$

Observe that

$$(5.5) \quad \sum_{F \in \mathcal{F}_{r,d}} \sum_{(\mathbf{a},x) \in Z(F)} 1 = \sum_{(\mathbf{a},x) \in \mathbb{F}_q^r} |\{F \in \mathcal{F}_{r,d} : F(\mathbf{a},x) = 0\}| = q^{\dim \mathcal{F}_{r,d} + r - 1}$$

Further, the function $h : (0, +\infty) \rightarrow \mathbb{R}$, $h(x) := \log x/x$ is increasing in the interval $[e, +\infty)$ and convex in the interval $[e^{3/2}, +\infty)$. By Corollary 5.3, the probability of the set of $F \in \mathcal{F}_{r,d}$ having up to $e^{3/2} = 4.48 \dots$ vertical strips is $\mathcal{O}(q^{-1})$. Therefore,

$$\begin{aligned} (5.6) \quad H &= \frac{\sum_{F \in \mathcal{F}_{r,d}} \sum_{(\mathbf{a},x) \in Z(F)} 1}{|\mathcal{F}_{r,d}|} \frac{\sum_{F \in \mathcal{F}_{r,d}} \sum_{(\mathbf{a},x) \in Z(F)} \frac{\log(NS(F)N_{\mathbf{a}}(F))}{NS(F)N_{\mathbf{a}}(F)}}{\sum_{F \in \mathcal{F}_{r,d}} \sum_{(\mathbf{a},x) \in Z(F)} 1} \\ &\geq q^{r-1} h \left(\frac{\sum_{F \in \mathcal{F}_{r,d}} \sum_{(\mathbf{a},x) \in Z(F)} NS(F)N_{\mathbf{a}}(F)}{\sum_{F \in \mathcal{F}_{r,d}} \sum_{(\mathbf{a},x) \in Z(F)} 1} \right) (1 + \mathcal{O}(q^{-1})). \end{aligned}$$

Next we analyze the numerator

$$\mathcal{N} := \sum_{F \in \mathcal{F}_{r,d}} \sum_{(\mathbf{a},x) \in Z(F)} NS(F)N_{\mathbf{a}}(F)$$

in the argument of h in the last expression.

Lemma 5.5. *We have $\mathcal{N} = 2\mu_d q^{2r-2+\dim \mathcal{F}_{r,d}}(1 + \mathcal{O}(q^{-1}))$.*

Proof. For $F \in \mathcal{F}_{r,d}$ and $\mathbf{a} \in VS(F)$, we have

$$NS(F) = \left| \bigcup_{x \in \mathbb{F}_q} \{\mathbf{a} \in \mathbb{F}_q^{r-1} : F(\mathbf{a},x) = 0\} \right|, \quad N_{\mathbf{a}}(F) = |\{x \in \mathbb{F}_q : F(\mathbf{a},x) = 0\}|.$$

As a consequence,

$$\begin{aligned}
\mathcal{N} &= \sum_{F \in \mathcal{F}_{r,d}} \sum_{\substack{(\mathbf{a}, x) \in \mathbb{F}_q^r \\ F(\mathbf{a}, x) = 0}} \sum_{\substack{y \in \mathbb{F}_q \\ F(\mathbf{a}, y) = 0}} \left| \bigcup_{z \in \mathbb{F}_q} \{\mathbf{b} \in \mathbb{F}_q^{r-1} : F(\mathbf{b}, z) = 0\} \right| \\
&= \sum_{F \in \mathcal{F}_{r,d}} \sum_{\substack{(\mathbf{a}, x) \in \mathbb{F}_q^r \\ F(\mathbf{a}, x) = 0}} \sum_{\substack{y \in \mathbb{F}_q \\ F(\mathbf{a}, y) = 0}} \sum_{k=1}^q (-1)^{k-1} \sum_{\substack{\mathcal{Z}_k \subset \mathbb{F}_q \\ |\mathcal{Z}_k| = k}} |\{\mathbf{b} \in \mathbb{F}_q^{r-1} : F(\mathbf{b}, T)|_{\mathcal{Z}_k} \equiv 0\}| \\
&= \sum_{k=1}^q (-1)^{k-1} \sum_{\mathbf{a} \in \mathbb{F}_q^{r-1}} \sum_{x \in \mathbb{F}_q} \sum_{y \in \mathbb{F}_q} \sum_{\substack{\mathcal{Z}_k \subset \mathbb{F}_q \\ |\mathcal{Z}_k| = k}} \mathcal{N}_{\mathbf{a}, x, y, \mathcal{Z}_k},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{N}_{\mathbf{a}, x, y, \mathcal{Z}_k} &:= \sum_{\substack{F \in \mathcal{F}_{r,d} \\ F(\mathbf{a}, x) = F(\mathbf{a}, y) = 0}} |\{\mathbf{b} \in \mathbb{F}_q^{r-1} : F(\mathbf{b}, T)|_{\mathcal{Z}_k} \equiv 0\}| \\
&= \sum_{\mathbf{b} \in \mathbb{F}_q^{r-1}} |\{F \in \mathcal{F}_{r,d} : F(\mathbf{a}, x) = 0, F(\mathbf{a}, y) = 0, F(\mathbf{b}, T)|_{\mathcal{Z}_k} \equiv 0\}|.
\end{aligned}$$

Suppose that $k \leq d$. For $\mathbf{b} \neq \mathbf{a}$ and $x \neq y$, the equalities $F(\mathbf{a}, x) = 0, F(\mathbf{a}, y) = 0, F(\mathbf{b}, T)|_{\mathcal{Z}_k} \equiv 0$ are linearly-independent conditions on the coefficients of F . If $\mathbf{b} \neq \mathbf{a}$ and $x = y$, then we have $k + 1$ linearly-independent conditions. Finally, for $\mathbf{b} = \mathbf{a}$, the number of linearly-independent conditions depends on the size of the intersection $\{x, y\} \cap \mathcal{Z}_k$. It follows that

$$\mathcal{N}_{\mathbf{a}, x, y, \mathcal{Z}_k} = (q^{r-1} - 1) q^{\dim \mathcal{F}_{r,d-k-|\{x,y\}|}} + q^{\dim \mathcal{F}_{r,d-\min\{d+1, |\{x,y\} \cup \mathcal{Z}_k\}|}}.$$

Therefore, by elementary calculations we obtain

$$\begin{aligned}
\sum_{x \in \mathbb{F}_q} \sum_{y \in \mathbb{F}_q} \sum_{\substack{\mathcal{Z}_k \subset \mathbb{F}_q \\ |\mathcal{Z}_k| = k}} \mathcal{N}_{\mathbf{a}, x, y, \mathcal{Z}_k} &= (q^{r-1} - 1) \binom{q}{k} q^{\dim \mathcal{F}_{r,d-k}} \left(\frac{q^2 - q}{q^2} + \frac{q}{q} \right) (1 + \mathcal{O}(q^{1-r})) \\
&= \frac{2q-1}{q} (q^{r-1} - 1) \binom{q}{k} q^{\dim \mathcal{F}_{r,d-k}} (1 + \mathcal{O}(q^{1-r})).
\end{aligned}$$

Now assume that $k > d$. Then the condition $F(\mathbf{b}, T)|_{\mathcal{Z}_k} \equiv 0$ is equivalent to $F(\mathbf{b}, T) = 0$. Arguing as above, we deduce that

$$\sum_{x \in \mathbb{F}_q} \sum_{y \in \mathbb{F}_q} \sum_{\substack{\mathcal{Z}_k \subset \mathbb{F}_q \\ |\mathcal{Z}_k| = k}} \mathcal{N}_{\mathbf{a}, x, y, \mathcal{Z}_k} = \frac{2q-1}{q} (q^{r-1} - 1) \binom{q}{k} q^{\dim \mathcal{F}_{r,d-(d+1)}} (1 + \mathcal{O}(q^{1-r})).$$

Putting these equalities together and using (2.4), we obtain

$$\begin{aligned}
\mathcal{N} &= 2q^{2r-2+\dim \mathcal{F}_{r,d}} \frac{2q-1}{2q} (1 - q^{1-r}) \\
&\quad \left(\sum_{k=1}^d (-1)^{k-1} \binom{q}{k} q^{-k} + \sum_{k=d+1}^q (-1)^{k-1} \binom{q}{k} q^{-d-1} \right) (1 + \mathcal{O}(q^{1-r})) \\
&= 2\mu_d q^{2r-2+\dim \mathcal{F}_{r,d}} (1 + \mathcal{O}(q^{-1})).
\end{aligned}$$

This finishes the proof of the lemma. \square

Combining (5.6) with (5.5) and Lemma 5.5, it follows that

$$H \geq q^{r-1} h \left(\frac{2\mu_d q^{2r-2+\dim \mathcal{F}_{r,d}} (1 + \mathcal{O}(q^{-1}))}{q^{r-1+\dim \mathcal{F}_{r,d}}} \right) (1 + \mathcal{O}(q^{-1})).$$

In other words, we have the following result.

Theorem 5.6. *If H denotes the average entropy of the SVS algorithm, then*

$$H \geq \frac{1}{2\mu_d} \log(q^{r-1})(1 + \mathcal{O}(q^{-1})).$$

Recall that, according to (5.3), for an algorithm for which the outputs are equidistributed we have the upper bound $H \leq \log(q^{r-1})$. For large d we have

$$\frac{1}{2\mu_d} \approx \frac{1}{2(1 - e^{-1})} \approx 0.79.$$

We may therefore paraphrase Theorem 5.6 as saying that the SVS algorithm is at least 79 per cent as good as any “ideal” algorithm.

6. SIMULATIONS ON TEST EXAMPLES

We end the paper with a description of the results on the number of searches that were obtained by executing the SVS algorithm on random samples of elements $\mathcal{F}_{r,d}$, for given values of q , r and d . Recall that $C : \mathbb{F} \times \mathcal{F}_{r,d} \mapsto \mathbb{N} \cup \{\infty\}$ denotes the random variable which counts the number of searches that are performed for all possible choices of vertical strips. Theorem 4.3 shows that

$$P[C = s] \approx (1 - \mu_d)^{s-1} \mu_d.$$

The simulations we exhibit were aimed to test whether the right-hand side of the previous expression approximates the left-hand side on the examples considered. For a random sample $\mathcal{S} \subset \mathcal{F}_{r,d}$ and $\underline{a} \in \mathbb{F}_s$, we use the following notations:

$$p_{\underline{a}} := p_{r,d}[\mathcal{S} \cap C_{\underline{a}} = s], \quad \hat{p}_s := (1 - \mu_d)^{s-1} \mu_d.$$

We take $N := 30$ choices of $\underline{a} \in \mathbb{F}_s$, and compute the sample mean

$$\bar{p}_s := \sum_{i=1}^N \frac{p_{\underline{a}_i}}{N}.$$

Furthermore, we consider the corresponding relative errors:

$$\epsilon_s := \frac{|\bar{p}_s - \hat{p}_s|}{\hat{p}_s}.$$

Finally, we compare the average number $\bar{N}_{r,d}^q$ of vertical strips searched with its theoretical upper bound according to Theorem 4.4, namely $1/\mu_d$.

We consider only relatively moderate values of s , since for higher values the probability $p_{\underline{a}}$ is so small that the corresponding information becomes uninteresting. This also explains the fact that relative errors ϵ_s tend to grow as s grows. Finally, we remark that, although polynomials without \mathbb{F}_q -rational zeros occur in some of the experiments described below, the number of such polynomial is so small that it does not affect the average behavior of our simulations.

6.1. Examples with $r := 2$ and $q := 67$ and $q := 8$. In this section we consider random samples of bivariate polynomials with coefficients in the finite field \mathbb{F}_{67} . In Table 1 we consider a random sample \mathcal{S} of 1000000 polynomials of $\mathbb{F}_{67}[X_1, X_2]$ of degree at most $d := 30$ and analyze how many vertical strips are searched on this sample. Therefore, we have $\hat{p}_s := (1 - \mu_{30})^{s-1} \mu_{30}$, where $\mu_{30} := 0.6321205588 \dots$. Further, we have $\bar{N}_{2,30}^{67} = 1.574924 \dots$, to be compared with $1/\mu_{30} = 1.581977 \dots$.

Our second example concerns a sample 1000000 polynomials of $\mathbb{F}_{67}[X_1, X_2]$ of degree at most $d := 5$. We have $\hat{p}_s := (1 - \mu_5)^{s-1} \mu_5$, where $\mu_5 := 0.6333333 \dots$. The corresponding results are summarized in Table 2. We observe that $\bar{N}_{2,5}^{67} = 1.572816 \dots$, to be compared with $1/\mu_5 = 1.578947 \dots$.

We end this section by considering polynomials with coefficients in a non-prime field, namely $\mathbb{F}_8[X_1, X_2]$. In this case, $\hat{p}_s := (1 - \mu_3)^{s-1} \mu_3$, where $\mu_3 := 0.666666 \dots$. In Table 3 the results for a sample of 100000 polynomials of degree at most $d := 3$ are exhibited. We have $\bar{N}_{3,3}^8 = 1.504512 \dots$, to be compared with $1/\mu_3 = 1.5$.

TABLE 1. Random sample with $q = 67$, $r = 2$ and $d = 30$.

s	\overline{p}_s	\widehat{p}_s	ϵ_s
1	0.635031	0.632121	0.004583
2	0.231664	0.232544	0.003799
3	0.084627	0.085548	0.010889
4	0.030921	0.031471	0.017789
5	0.011279	0.011578	0.026473
6	0.004101	0.004259	0.038575
7	0.001509	0.001567	0.038166
8	0.000553	0.000576	0.042349
9	0.000199	0.000212	0.067918
10	0.000076	0.000078	0.030513
11	0.000025	0.000029	0.161872
12	0.000010	0.000011	0.038441
13	0.000038	0.000003	0.022074
14	0.000011	0.000001	0.339501
15	0.000001	0.000001	0.051253

TABLE 2. Random sample with $q = 67$, $r = 2$ and $d = 5$.

s	\overline{p}_s	\widehat{p}_s	ϵ_s
1	0.635885	0.633333	0.004012
2	0.231459	0.232222	0.003298
3	0.084318	0.085148	0.009844
4	0.030727	0.031221	0.016085
5	0.011188	0.011448	0.023224
6	0.004091	0.004197	0.025996
7	0.001481	0.001539	0.039029
8	0.000543	0.000564	0.040109
9	0.000195	0.000207	0.056976
10	0.000069	0.000076	0.085938
11	0.000029	0.000028	0.030685
12	0.000009	0.000010	0.129198
13	0.000003	0.000003	0.133380
14	0.000002	0.000001	0.085740
15	0.000001	0.000001	0.057169

TABLE 3. Random sample with $q = 8$, $r = 3$ and $d = 3$.

s	\overline{p}_s	\widehat{p}_s	ϵ_s
1	0.663161	0.666666	0.005259
2	0.222801	0.222222	0.002605
3	0.075617	0.074074	0.014151
4	0.025319	0.024691	0.020831
5	0.008725	0.008230	0.060146
6	0.002859	0.002743	0.042289

6.2. **Examples with $r := 3$ and $q := 11$ and $q := 67$.** Finally, we consider two samples of 1000000 polynomials of $\mathbb{F}_q[X_1, X_2, X_3]$. The first sample contains polynomials of degree at most $d := 5$ with coefficients in \mathbb{F}_{11} , while the second one contains polynomials of degree at most $d := 5$ with coefficients in \mathbb{F}_{67} . Results are exhibited in Tables 4 and 5 respectively. The average numbers of searched vertical

strips are $\overline{N}_{3,5}^{11} = 1.539646\dots$ and $\overline{N}_{3,5}^{67} = 1.572975\dots$, both to be compared with $1/\mu_5 = 1.578947\dots$.

TABLE 4. Random sample with $q = 11$, $r = 3$ and $d = 5$.

s	\overline{p}_s	\widehat{p}_s	ϵ_s
1	0.649494	0.633333	0.024881
2	0.227637	0.232222	0.020145
3	0.079769	0.085148	0.067430
4	0.027999	0.031221	0.115075
5	0.009822	0.011448	0.165519
6	0.003419	0.004198	0.227683
7	0.001213	0.001539	0.269344
8	0.000421	0.000564	0.340555
9	0.000149	0.000207	0.382851
10	0.000050	0.000076	0.504379
11	0.000017	0.000028	0.662509
12	0.000002	0.000010	0.500062
13	0.000002	0.000004	0.726225
14	0.000001	0.000001	0.523767
15	0.000000	0.000001	2.017058

TABLE 5. Random sample with $q = 67$, $r = 3$ and $d = 5$.

s	\overline{p}_s	\widehat{p}_s	ϵ_s
1	0.635802	0.633333	0.003883
2	0.231571	0.232222	0.002810
3	0.084285	0.085148	0.010237
4	0.030732	0.031221	0.015898
5	0.011192	0.011447	0.022809
6	0.004081	0.004197	0.028645
7	0.001482	0.001539	0.038865
8	0.000541	0.000564	0.042865
9	0.000199	0.000207	0.039628
10	0.000071	0.000076	0.062618
11	0.000027	0.000028	0.017780
12	0.000010	0.000010	0.003320
13	0.000003	0.000004	0.078891
14	0.000001	0.000001	0.111938
15	0.000000	0.000001	0.257107

Summarizing, the results of Tables 1–5 show that the behavior predicted by the asymptotic estimates of Theorems 4.3 and 4.4 is also appreciated in the numerical experiments we perform. Nevertheless, as the cost of the SVS algorithm grows exponentially with the number r of variables under consideration, our experiments only considered the cases $r = 2$ and $r = 3$.

ACKNOWLEDGEMENTS

The authors gratefully acknowledge the comments by the anonymous referees, which helped to significantly improve the presentation of the results of this paper.

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